

# The isomorphism type of the centralizer of an element in a Lie group

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## Abstract

Let  $G$  be an 1-connected simple Lie group, and let  $x \in G$  be a group element. We determine the isomorphism type of the centralizer  $C_x$  in term of a minimal geodesic joining the group unit  $e \in G$  to  $x$ .

This result is applied to classify the isomorphism types of maximal subgroups of maximal rank of  $G$  [4], and the isomorphism types of parabolic subgroups of  $G$ .

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## 1 Introduction

Let  $G$  be a compact connected semisimple Lie group with a given element  $x \in G$ . The *centralizer*  $C_x$  of  $x$  and the *adjoint orbit*  $M_x$  through  $x$  are the subspaces of  $G$

$$C_x = \{g \in G \mid gx = xg\}, \quad M_x = \{gxg^{-1} \in G \mid g \in G\},$$

respectively. The map  $G \rightarrow G$  by  $g \rightarrow gxg^{-1}$  is constant along the left cosets of  $C_x$  in  $G$ , and induces a diffeomorphism from the *homogeneous space*  $G/C_x$  onto the orbit space  $M_x$

$$f_x : G/C_x \xrightarrow{\cong} M_x \text{ by } [g] \rightarrow gxg^{-1}.$$

The manifolds  $G/C_x$  arising in this fashion have offered many important subjects in geometry, as shown in the next two examples.

**Example 1.1.** Let  $Z(G)$  be the center of  $G$  and let  $x \in G$  be an element with  $x^r \in Z(G)$  for some power  $r \geq 2$ . The homogeneous space  $G/C_x$  possesses a canonical periodic  $r$  automorphism

$$\sigma_x : G/C_x \rightarrow G/C_x \text{ by } \sigma_x[g] = [xgx^{-1}].$$

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For this reason the pair  $(G/C_x, \sigma_x)$  is called an  $r$ -symmetric space of  $G$  in [8]. In the special case of  $r = 2$ , they are the global Riemannian symmetric spaces of  $G$  in the sense of E. Cartan [11].  $\square$

**Example 1.2.** If the minimal geodesic joining the group unit  $e \in G$  to  $x$  is unique, the centralizer  $C_x$  is a *parabolic subgroup* of  $G$ . The corresponding homogeneous space  $G/C_x$  is called a *flag manifold* of  $G$ , which is the focus of the classical Schubert calculus [5, 9, 10, 13, 14].  $\square$

To investigate the geometry and topology of the homogeneous space  $G/C_x$  it is often necessary to determine explicitly the isomorphism type of the centralizer  $C_x$  in term of  $x \in G$ . However, in the existing literatures one merely finds method to decide its local type in some special cases [11, 4, 16], see Remark 2.10. The purpose of this paper is to give an explicit procedure for calculating the centralizer  $C_x$  in term of a minimal geodesic joining the unit  $e$  to  $x$ , see Theorem 4.3 in §4.1. This result is applied to classify the isomorphism types of maximal subgroups of maximal rank of  $G$  in §4.2, and of parabolic subgroups of  $G$  in §4.3.

To be precise some notation are needed. For a compact connected Lie group  $K$  the identity component of the center  $\mathcal{Z}(K)$  of  $K$  will be denoted by  $K^{Rad}$ , and will be called *the radical part* of  $K$  (this is always a connected torus subgroup of  $K$ ). According to Cartan's classification on compact Lie groups, up to isomorphism, the group  $K$  admits a canonical presentation of the form

$$(1.1) \quad K \cong (G_1 \times \cdots \times G_k \times K^{Rad})/H$$

in which

- i) each  $G_t$  is one of the 1-connected *simple Lie groups*,  $1 \leq t \leq k$ ;
- ii) the denominator  $H$  is a finite subgroup of  $\mathcal{Z}(G_1) \times \cdots \times \mathcal{Z}(G_k) \times K^{Rad}$ .

It is also known that all 1-connected simple Lie groups  $G$ , together with their centers, are classified by the types  $\Phi_G$  of their corresponding root systems tabulated below [12, p.57]

$G$	$SU(n)$	$Sp(n)$	$Spin(2n+1)$	$Spin(2n)$	$G_2$	$F_4$	$E_6$	$E_7$	$E_8$
$\Phi_G$	$A_{n-1}$	$B_n$	$C_n$	$D_n$	$G_2$	$F_4$	$E_6$	$E_7$	$E_8$
$\mathcal{Z}(G)$	$Z_n$	$Z_2$	$Z_2$	$Z_4, n = 2k+1$ $Z_2 \oplus Z_2, n = 2k$	$\{e\}$	$\{e\}$	$Z_3$	$Z_2$	$\{e\}$

Table 1. The types and centers of 1-connected simple Lie groups

**Definition 1.3.** In the presentation (1.1) the group  $G_1 \times \cdots \times G_k$  is called the *semisimple part* of  $K$ , and is denoted by  $K^s$ .

The obvious quotient (i.e. covering) map  $\pi : K^s \times K^{Rad} \rightarrow K$  is called *the local type* of  $K$ .  $\square$

**Corollary 1.4.** Let  $K$  be a compact connected Lie group. The following statements are equivalent.

- i) The group  $K$  is semisimple;
- ii)  $K^{Rad} = \{e\}$ ;
- iii) The local type  $\pi$  of  $K$  agrees with the universal cover of  $K$ .  $\square$

The paper is arranged as the following. Section 2 contains a brief introduction on the roots and weight systems of Lie groups, and obtains the local type  $\pi : C_x^s \times C_x^{Rad} \rightarrow C_x$  of a centralizer  $C_x$  in term of  $x \in G$  in Theorem 2.8. In section 3 we introduce for each compact connected Lie group  $K$  so called *extended weight lattice*  $\Pi_K^0$ , together with two *deficiency functions* on it. They are applied in Theorem 3.7 to specify the isomorphism type of a subgroup  $K$  in a semisimple Lie group  $G$ . Summarizing results in Theorems 2.8 and 3.7, an explicit procedure for calculating the isomorphism type of a centralizer  $C_x$  in an 1-connected Lie group  $G$  is given in Theorem 4.3 of Section §4.1. Finally, to demonstrate the use of Theorem 4.3 we determine in §4.2 and §4.3 all the centralizers  $C_{\exp(u)}$  in an 1-connected exceptional Lie group  $G$  with  $u$  a multiple of a fundamental dominant weight of  $G$ .

## 2 The local type of a centralizer

In this paper  $K$  denotes a compact connected Lie group, and the notion  $G$  is reserved for the compact semisimple ones.

Equip the Lie algebra  $L(K)$  of  $K$  with an inner product  $(,)$  so that the adjoint representation acts as isometries on  $L(K)$ . Fixing a maximal torus  $T$  on  $K$  the *Cartan subalgebra* of  $K$  is the linear subspace  $L(T)$  of  $L(K)$ . The dimension  $n = \dim T$  is called the *rank* of the group  $K$ .

### 2.1 The root system of a compact connected Lie group

The restriction of the exponential map  $\exp : L(K) \rightarrow K$  to the subspace  $L(T)$  defines a set  $\mathcal{S}(K) = \{L_1, \dots, L_m\}$  of  $m = \frac{1}{2}(\dim K - n)$  hyperplanes in  $L(T)$ , namely, the set of *singular hyperplanes* through the origin in  $L(T)$  [2, p.168]. Let  $l_k \subset L(T)$  be the normal line of the plane  $L_k$  through the origin,  $1 \leq k \leq m$ . Then the map  $\exp$  carries  $l_k$  onto a circle subgroup on  $T$ . Let  $\pm\alpha_k \in l_k$  be the non-zero vectors with minimal length so that  $\exp(\pm\alpha_k) = e$ ,  $1 \leq k \leq m$ .

**Definition 2.1.** The subset  $\Phi_K = \{\pm\alpha_k \in L(T) \mid 1 \leq k \leq m\}$  of  $L(T)$  is called the *root system* of  $K$ .  $\square$

**Remark 2.2.** We note that the root system  $\Phi_K$  by Definition 2.1 is *dual* to those that are commonly used in literatures, e.g. [1, 12]. In particular, the symplectic group  $Sp(n)$  is of the type  $B_n$ , while the spinor group  $Spin(2n+1)$  is of the type  $C_n$ .  $\square$

The planes in  $\mathcal{S}(K)$  divide  $L(T)$  into finitely many convex regions, called the *Weyl chambers* of  $K$ . Fix a regular point  $x_0 \in L(T)$ , and let  $\mathcal{F}(x_0)$  be the closure of the Weyl chamber containing  $x_0$ . Assume that  $L(x_0) = \{L_1, \dots, L_h\}$  is the subset of  $\mathcal{S}(K)$  consisting of the walls of  $\mathcal{F}(x_0)$ . Then

(2.1)  $h \leq n$ , where the equality holds if and only if  $K$  is semi-simple.

Let  $\alpha_i \in \Phi_K$  be the root normal to the wall  $L_i \in L(x_0)$  and pointing toward  $x_0$ .

**Definition 2.3.** The subset  $S(x_0) = \{\alpha_1, \dots, \alpha_h\}$  of the root system  $\Phi_K$  is called the *system of simple roots* of  $K$  relative to  $x_0$ .

The *Cartan matrix* of  $K$  (relative to  $x_0$ ) is the  $h \times h$  matrix defined by

$$(2.2) \quad A = (b_{ij})_{h \times h}, \quad b_{ij} = 2(a_i, \alpha_j)/(\alpha_j, \alpha_j).$$

The lattice in  $L(T)$  spanned by all simple roots is called the *root lattice*, and is denoted by  $\Lambda_K^r$ . The subset of  $\Lambda_K^r$  consisting of the sums of the simple roots  $\alpha_1, \dots, \alpha_h$  is denoted by  $\Lambda_K^{r,+}$ . We shall also put

$$\Phi_K^+ = \Lambda_K^{r,+} \cap \Phi_K,$$

whose elements are called the *positive roots* of  $K$ .  $\square$

The set  $S(x_0)$  of simple roots defines a partial order  $\prec$  on  $L(T)$  by the following rule: for two vectors  $u, v \in L(T)$  we say  $v \prec u$  if and only if the difference  $u - v$  is a sum of elements in  $S(x_0)$  (i.e. belongs to  $\Lambda_K^{r,+}$  [12, p.47]).

If  $G$  is a simple Lie group, elements in  $\Phi_G$  has at most two lengths. Let  $\beta \in \Phi_G^+$  (resp.  $\gamma \in \Phi_G^+$ ) be the unique *maximal short root* (resp. unique *maximal long root*) relative to the partial order  $\prec$  on  $\Phi_G^+$ . From [12, p.66, Table 2] one gets

**Lemma 2.4.** *Let  $G$  be a simple Lie group. We have  $\beta = \gamma$  unless  $G = G_2, F_4, B_n$  or  $C_n$ .*

Moreover, if  $G = G_2, F_4, B_n$  or  $C_n$  we have  $\beta \prec \gamma$  and the lengths of the three vectors  $\gamma, \beta, \delta = \gamma - \beta$  are given in the table below

	$\ \gamma\ ^2$	$\ \beta\ ^2$	$\ \delta\ ^2$
$G_2$	6	2	2
$F_4$	2	1	1
$B_n$	2	1	1
$C_n$	4	2	2

## 2.2 The weight system of a semisimple Lie group

A nonzero vector  $\alpha \in \Lambda_K^r$  gives rise to a linear map

$$(2.3) \quad \alpha^* : L(T) \rightarrow \mathbb{R} \text{ by } \alpha^*(x) = 2(x, \alpha)/(\alpha, \alpha).$$

If  $\alpha \in \Phi_K$  is a root, the map  $\alpha^*$  is called the *inverse root* of  $\alpha$  ([12, p.67]).

**Definition 2.5.** Assume that  $G$  is a semisimple Lie group. The *weight lattice* of  $G$  is the subset of  $L(T)$

$$\Lambda_G = \{x \in L(T) \mid \alpha^*(x) \in \mathbb{Z} \text{ for all } \alpha \in \Phi_G\},$$

whose elements are called *weights*. Its subset

$$(2.4) \quad \Omega_G = \{\omega_i \in L(T) \mid \alpha_j^*(\omega_i) = \delta_{i,j}, \alpha_j \in S(x_0)\}$$

is called the set of *fundamental dominant weights* of  $G$  relative to  $x_0$ , where  $\delta_{i,j}$  is the Kronecker symbol.  $\square$

**Lemma 2.6.** *Let  $G$  be a semisimple Lie group with Cartan matrix  $A$ , and let  $\Omega_G = \{\omega_1, \dots, \omega_n\}$  be the set of fundamental dominant weights relative to the regular point  $x_0$ . Then*

- i)  $\Omega_G = \{\omega_1, \dots, \omega_n\}$  is a basis for  $\Lambda_G$  over  $\mathbb{Z}$ ;
- ii) the fundamental dominant weights  $\omega_1, \dots, \omega_n$  can be expressed in term of the simple roots  $\{\alpha_1, \dots, \alpha_n\}$  as

$$(2.5) \quad \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix};$$

iii) for each  $1 \leq i \leq n$  the half line  $\{t\omega_i \in L(T) \mid t \in \mathbb{R}^+\}$  is the edge of the Weyl chamber  $\mathcal{F}(x_0)$  opposite to the wall  $L_i$ .  $\square$

**Proof.** With the assumption that  $G$  is semisimple, we have that  $h = n$  by (2.1), and that the Weyl chamber  $\mathcal{F}(x_0)$  is a convex cone with vertex  $0 \in L(T)$ .

Properties i) and ii) are known. By (2.4) each weight  $\omega_i \in \Omega_G$  is perpendicular to all the roots  $\alpha_j$  (i.e.  $\omega_i \in L_j$ ) with  $j \neq i$ . This shows iii).  $\square$

For a simple Lie group  $G$  we shall adopt the convention that its fundamental dominant weights  $\omega_1, \dots, \omega_n$  are ordered by the order of their corresponding simple roots pictured as the vertices in the Dynkin diagram of  $G$  [12, p.58]. Let  $\Pi_G \subseteq \Omega_G = \{\omega_1, \dots, \omega_n\}$  be the subset of *minimal weights* with respect to the partial order  $\prec$  (see §2.1) on  $\Omega_G$ . By considering the center  $\mathcal{Z}(G)$  as a finite subgroup of  $T$  one has the relation

$$(2.6) \quad \exp(\Pi_G \sqcup \{0\}) = \mathcal{Z}(G), \text{ see [12, p.72, Exercise 13].}$$

Explicitly, for each simple Lie group  $G$  with type  $\Phi_G$  the set  $\Pi_G$  of minimal weights is presented in Table 2 below:

$\Phi_G$	$A_n$	$B_n$	$C_n$	$D_n$	$E_6$	$E_7$
$\Pi_G$	$\{\omega_i\}_{1 \leq i \leq n}$	$\{\omega_n\}$	$\{\omega_1\}$	$\{\omega_1, \omega_{n-1}, \omega_n\}$	$\{\omega_1, \omega_6\}$	$\{\omega_7\}$

Table 2. The set  $\Pi_G$  of minimal weights of a simple Lie group  $G$

### 2.3 Computing in the fundamental Weyl cell

Let  $G$  be an 1-connected simple Lie group with maximal short root  $\beta \in \Phi_G^+$ . The *fundamental Weyl cell* of  $G$  is the simplex in  $\mathcal{F}(x_0)$  defined by

$$\Delta = \{u \in \mathcal{F}(x_0) \mid \beta^*(u) \leq 1\}.$$

In view of property iii) of Lemma 2.6, a vector  $u \in \Delta$  if and only if there is a subset  $I_u = \{k_1, \dots, k_r\} \subseteq \{1, \dots, n\}$  so that

$$(2.7) \quad u = \lambda_{k_1} \omega_{k_1} + \dots + \lambda_{k_r} \omega_{k_r} \text{ with } \lambda_{k_s} > 0 \text{ and } \beta(u) \leq 1.$$

Let  $\bar{I}_u$  be the complement of  $I_u$  in  $\{1, \dots, n\}$ .

**Lemma 2.7.** *If  $u \in \Delta$  is nonzero with  $u \notin \Omega_G$  one has*

$$(2.8) \quad 0 \leq \alpha^*(u) \leq 1 \text{ for any positive root } \alpha \in \Phi_G^+$$

Moreover

- i)  $\alpha^*(u) = 0$  implies that  $\alpha$  is a sum of the simple roots  $\alpha_i$  with  $i \in \bar{I}_u$ ;
- ii)  $\alpha^*(u) = 1$  implies that  $\beta^*(u) = 1$ , and that there is an  $k \in \{1, 2\}$  so that  $k\beta - \alpha$  is a sum of the simple roots  $\alpha_i$  with  $i \in \bar{I}_u$ .

**Proof.** Let  $d : T \times T \rightarrow \mathbb{R}$  be the distance function on  $T$  induced from the metric on  $L(G)$ . Since  $G$  is 1-connected,  $u \in \Delta$  implies that  $d(e, \exp(u)) = \|u\|$ , see [6, 7]. It follows that  $\|u\| \leq \|u - \alpha\|$  for any  $\alpha \in \Lambda_G^+$ . In particular,

$$(2.9) \quad \alpha^*(u) = \frac{2(\alpha, u)}{(\alpha, \alpha)} \leq 1 \text{ for all nonzero } \alpha \in \Lambda_G^{r,+}.$$

On the other side, express  $\alpha \in \Lambda_G^{r,+}$  in term of the simple roots as

$$\alpha = k_1\alpha_1 + \cdots + k_n\alpha_n \text{ with } k_i \in \mathbb{Z}^+.$$

With respect to the expression (2.7) we have

$$(2.10) \quad \alpha^*(u) = \frac{2}{(\alpha, \alpha)} \left( \sum_{i \in I_u} \lambda_i k_i (\alpha_i, \omega_i) \right) = \frac{1}{(\alpha, \alpha)} \left( \sum_{i \in I_u} \lambda_i k_i (\alpha_i, a_i) \right) \geq 0$$

Since  $\Phi_G^+ \subset \Lambda_G^{r,+}$  the relation (2.8) has been shown by (2.9) and (2.10).

By (2.10)  $\alpha^*(u) = 0$  implies that  $k_i = 0$ ,  $i \in I_u$ . This shows i).

Let  $\alpha \in \Phi_G^+$  be with  $\alpha^*(u) = 1$ . The proof of ii) will be divided into two cases, depending on whether  $\alpha$  is a short or a long root.

**Case 1.** If  $\alpha$  is short we get from  $\alpha \prec \beta$  that

$$\beta - \alpha = k_1\alpha_1 + \cdots + k_n\alpha_n \text{ with } k_i \in \mathbb{Z}^+.$$

Consequently,

$$1 \geq \beta^*(u) = \alpha^*(u) + \frac{1}{(\beta, \beta)} \left( \sum_{i \in I_u} \lambda_i k_i (\alpha_i, a_i) \right) \geq 1,$$

where the first inequality  $\geq$  comes from (2.8), and the second follows from

$$\alpha^*(u) = 1 \text{ and } \frac{1}{(\beta, \beta)} \left( \sum_{i \in I_u} \lambda_i k_i (\alpha_i, a_i) \right) \geq 0.$$

This is possible unless  $k_i = 0$  for all  $i \in I_u$ . This shows ii) when  $\alpha$  is short.

**Case 2.** If  $\alpha$  is long we get from  $\alpha \prec \gamma$  that

$$\gamma - \alpha = k_1\alpha_1 + \cdots + k_n\alpha_n \text{ with } k_i \in \mathbb{Z}^+.$$

The relation

$$1 \geq \gamma^*(u) = \alpha^*(u) + \frac{1}{(\gamma, \gamma)} \left( \sum_{i \in I_u} \lambda_i k_i (\alpha_i, a_i) \right) \geq 1$$

and the assumption  $\alpha^*(u) = 1$  force that

$$(2.11) \quad \gamma^*(u) = 1 \text{ and } \gamma - \alpha = \sum_{i \in I_u} k_i \alpha_i, \quad k_i \in \mathbb{Z}^+.$$

Write  $\gamma = \beta + \delta$  with  $\delta = \gamma - \beta$  (note that  $\delta \in \Lambda_G^{r,+}$ ). From  $\gamma^*(u) = 1$  and  $\|\beta\|^2 = \|\delta\|^2$  by Lemma 2.4 we get that

$$\frac{\|\gamma\|^2}{\|\beta\|^2} = \beta^*(u) + \delta^*(u),$$

where  $\beta^*(u), \delta^*(u) \leq 1$  by (2.9). Again by Lemma 2.4 this is possible if and only if when  $G = F_4, B_n, C_n$  and

$$(2.12) \quad \gamma^*(u) = \beta^*(u) = \delta^*(u) = 1.$$

Assume, apart from (2.7), that

$$u = \lambda_1 \omega_1 + \cdots + \lambda_n \omega_n.$$

If  $G = F_4$  the system (2.12) becomes

$$\begin{aligned} 2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 &= 2\lambda_1 + 4\lambda_2 + 3\lambda_3 + 2\lambda_4 \\ &= 2\lambda_1 + 2\lambda_2 + \lambda_3 = 1. \end{aligned}$$

It implies that  $\lambda_1 = \frac{1}{2}$ ;  $\lambda_2 = \lambda_3 = \lambda_4 = 0$  and consequently,  $\bar{T}_u = \{2, 3, 4\}$ . The proof of ii) for  $G = F_4$  is completed by (2.11) and

$$\gamma = 2\beta - \alpha_2 - 2\alpha_3 - 2\alpha_4 \text{ (see [12, p.66, Table 2])}.$$

If  $G = B_n$  the system (2.12) gives

$$\begin{aligned} \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + \lambda_n &= 2\lambda_1 + \cdots + 2\lambda_{n-1} + \lambda_n \\ &= 2\lambda_2 + \cdots + 2\lambda_{n-1} + \lambda_n = 1 \end{aligned}$$

It implies that  $\lambda_1 = 0$  and consequently  $\bar{T}_u \supseteq \{1\}$ . The proof of ii) for  $G = B_n$  is completed by (2.11) and

$$\gamma = 2\beta - \alpha_1 \text{ (see [12, p.66, Table 2])}.$$

Finally, if  $G = C_n$  the system (2.12) turns to be

$$\lambda_1 + \cdots + \lambda_n = \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_n = \lambda_1 = 1.$$

It implies that  $\lambda_1 = 1$ ,  $\lambda_2 = \lambda_3 = \lambda_4 = 0$  and consequently  $u = \omega_1$ . This contradiction to  $u \notin \Omega_G$  completes the proof of ii) for  $G = C_n$ .  $\square$

## 2.4 The local type of a centralizer

Let  $G$  be an 1-connected simple Lie group with maximal short root  $\beta \in \Phi_G^+$  and Dynkin diagram  $\Gamma_G$ . The extended Dynkin diagram of  $G$  with respect to  $-\beta$  is denoted by  $\tilde{\Gamma}_G$ . For a  $u \in \Delta$  given as that in (2.7) denote by  $T_u$  (resp.  $T_u^\beta$ ) for the identity component of the subgroup of  $T$ :

$$\bigcap_{i \in \bar{T}_u} \ker[\tilde{\alpha}_i : T \rightarrow S^1] \text{ (resp. } \ker[\tilde{\beta} : T \rightarrow S^1] \bigcap_{i \in \bar{T}_u} \ker[\tilde{\alpha}_i : T \rightarrow S^1]),$$

where  $S^1$  is the circle group  $\{\exp(2\pi it) \in \mathbb{C} \mid t \in [0, 1]\}$ , and where  $\tilde{\alpha} : T \rightarrow S^1$  is the homomorphism whose tangent map at the group unit is the inverse root  $\alpha^* : L(T) \rightarrow \mathbb{R}$  of  $\alpha \in \Phi_G$  (i.e. (2.3)). Let  $\Gamma_u \subseteq \Gamma_G$  (resp.  $\Gamma_u^\beta \subseteq \tilde{\Gamma}_G$ ) be the subdiagram obtained by deleting all the vertices  $\alpha_k$  with  $k \in I_u$ , as well as the edges adjoining to it, from  $\Gamma_G$  (resp.  $\tilde{\Gamma}_G$ ).

Since each  $x \in G$  is conjugate in  $G$  to an element of the form  $\exp(u)$  with  $u \in \Delta$ , and since the isomorphism type of a subgroup of  $G$  remains invariant under conjugation, the study the isomorphism type of a centralizer  $C_x$ ,  $x \in G$ , can be reduced to the cases  $x = \exp(u)$ ,  $u \in \Delta$ . Geometrically, the path  $\gamma_u(t) = \exp(tu)$ ,  $t \in [0, 1]$ , is a minimal geodesic on  $G$  joining the unit  $e$  to  $x$ .

Let  $C_x^s$  and  $C_x^{Rad}$  be respectively the semisimple part and the radical part of the centralizer  $C_x$ . In view of the fact that the semisimple part of a group

is classified by its Dynkin diagram [12, p.56], the next result specifies the local type of the centralizer  $C_{\exp(u)}$  in term of  $u \in \Delta$ .

**Theorem 2.8.** *Let  $G$  be a compact and 1-connected Lie group and let  $x = \exp(u) \in G$  be with  $u \in \Delta$  but  $u \notin \Omega_G \sqcup \{0\}$ . Then the centralizer  $C_x$  is a compact, connected and proper subgroup of  $G$ .*

Moreover,

- i) if  $\beta(u) < 1$ , then  $\Gamma_{C_x^s} = \Gamma_u$ ,  $C_x^{Rad} = T_u$ ;
- ii) if  $\beta(u) = 1$ , then  $\Gamma_{C_x^s} = \Gamma_u^\beta$ ,  $C_x^{Rad} = T_u^\beta$ .

**Proof.** According to Borel [3, Corollary 3.4, p.101] the centralizer  $C_x$  in an 1-connected Lie group  $G$  is always connected. Furthermore, with the assumption that  $u \notin \Omega_G \sqcup \{0\}$ , the group  $C_x$  must be a proper subgroup of  $G$ .

To show i) and ii) assume that the Cartan decomposition of  $L(G)$  is

$$L(G) = L(T) \oplus \bigoplus_{\alpha \in \Phi_G^+} L_\alpha,$$

where  $L_\alpha$  is the root space belonging to the root  $\alpha \in \Phi_G^+$  ([12, p.35]). According to [2, p.189], for  $x = \exp(u)$  with  $u \in L(T)$  the Cartan decomposition of the Lie algebra  $L(C_x)$  is

$$(2.13) \quad L(C_x) = L(T) \oplus \bigoplus_{\alpha \in \Psi_u} L_\alpha, \text{ where } \Psi_u = \{\alpha \in \Phi_G^+ \mid \alpha^*(u) \in \mathbb{Z}\}.$$

In view of (2.13) the set  $\Phi_u = \{\pm\alpha \mid \alpha \in \Psi_u\}$  can be identified with the root system of the semisimple part  $C_x^s$  of  $C_x$ .

If  $u \in \Delta$  with  $\beta(u) < 1$  we have by Lemma 2.7 that  $\Psi_u = \{\alpha \in \Phi_G^+ \mid \alpha^*(u) = 0\}$  and that

- a) the set of simple roots  $\alpha_i$  with  $i \in \bar{T}_u$  is a base of  $\Phi_u$  [12, p.47].

Consequently, from the definition of the subgroup  $T_u$  one gets

- b)  $T_u \subseteq C_x^{Rad}$ .

The relation  $\Gamma_{C_x^s} = \Gamma_u$  is shown by a). For the dimension reason  $\dim T_u + \text{rank } C_x^s = \dim T$  we get from b) that  $T_u = C_x^{Rad}$ . This finishes the proof of i).

Similarly, if  $u \in \Delta$  with  $\beta(u) = 1$  we have by Lemma 2.7 that  $\Psi_u = \{\alpha \in \Phi_G^+ \mid \alpha^*(u) = 0, 1\}$  and that

- c) any element in  $\Phi_u$  is a linear combination of the simple roots  $\alpha_i$  with  $i \in \bar{T}_u$  and  $-\beta$  with coefficients all nonnegative or nonpositive.

As a result, we get from c) and the definition of the subgroup  $T_u^\beta \subset T$  that

- d)  $T_u^\beta \subseteq C_x^{Rad}$ .

With the assumptions  $\bar{T}_u \neq \emptyset$  and  $u \notin \Omega_G$ , we conclude by c) that the set  $\{\alpha_i, -\beta \mid i \in \bar{T}_u\}$  is a base of  $\Phi_u$  [12, p.47] and therefore,  $\Gamma_{C_x^s} = \Gamma_u^\beta$ . Again, for dimension reason we get from d) that  $T_u^\beta = C_x^{Rad}$ . This completes the proof.  $\square$

For an 1-connected simple Lie group  $G$  with rank  $n$  assume that the expression of the maximal short root  $\beta \in \Phi_G^+$  in terms of the simple roots is



$$\beta = m_1\alpha_1 + \cdots + m_n\alpha_n.$$

The set of vertices of the Weyl cell  $\Delta$  is clearly given by

$$\mathcal{V}_G = \{0, X_i = \frac{(\beta, \beta)}{m_i(\alpha_i, \alpha_i)}\omega_i \in \Delta \mid 1 \leq i \leq n\}.$$

Let us put  $\mathcal{F}_G = \{u_i \in \Delta \mid 1 \leq i \leq n\}$  with

$$u_i = \begin{cases} \frac{1}{2}X_i & \text{if } \alpha_i \text{ is short and } m_i = 1; \\ X_i & \text{otherwise.} \end{cases}$$

According to Theorem 2.8, for two vectors  $u, u' \in \Delta$  one has  $L(C_{\exp(u)}) \supseteq L(C_{\exp(u')})$  if either  $I_u \subseteq I_{u'}$  and  $\beta(u), \beta(u') < 1$ , or  $I_u = I_{u'}$  and  $\beta(u') \leq \beta(u) = 1$ . Since a maximal connected subgroup of maximal rank of  $G$  must be the centralizer of some element in  $G$  ([4, Theorem 5]), we get from Theorem 2.8 the classical result due to Borel and Siebenthal [4, §7]:

**Corollary 2.9.** *For an 1-connected simple Lie group  $G$  with rank  $n$ , the set of centralizers  $\{C_{\exp(u_i)} \mid 1 \leq i \leq n\}$  contains all the isomorphism types of maximal subgroups of maximal rank of  $G$ .  $\square$*

**Remark 2.10.** In the classical paper [4] Borel and Siebenthal intended to find all maximal subgroups of maximal rank of compact connected Lie groups. For the 1-connected simple Lie groups they give the answers only up to local types. As application of our Theorem 4.3, the isomorphism types of these groups will be determined in Theorem 4.4, compare Table 4 in §4.2 with the table in [4, §7].

In [16] M. Reeder gives a description of the Lie algebra  $L(C_{\exp(u)})$  under the assumption that  $m \cdot u \in \Lambda_G^e$  for some multiple  $m$ . We emphasize that Theorem 2.8 is not obvious, in view of the crucial use of Lemma 2.7 in its proof.

In a recent Web discussion J. Newman suggested the problem of finding an algorithm for computing the isomorphism type of the centralizer  $C_H$  of a finite subgroup  $H$  of a simple Lie group  $G$  [15]. If  $\{u_1, \dots, u_k\} \subset \Delta$  is a set of vectors in the cell  $\Delta$  and if  $H$  is the subgroup of  $G$  generated by  $\exp(u_i)$ ,  $1 \leq i \leq k$ . then the proof of Theorem 2.8, together with a comment of A. Knutson in the discussion [15], implies the next Cartan decomposition of the Lie algebra of  $C_H$

$$L(C_H) = L(T) \oplus \bigoplus_{\alpha(u_1)=\dots=\alpha(u_k)=0} L_\alpha \oplus \bigoplus_{\alpha(u_1)=\dots=\alpha(u_k)=1} L_\alpha, \alpha \in \Phi_G^+.$$

More precisely, a base for the root system of the group  $C_H$  is either

- i)  $\{\alpha_i, -\beta \mid i \in \bar{I}_{u_1} \cap \dots \cap \bar{I}_{u_k}\}$  if  $\beta(u_i) = 1$  for all  $i$ , or
- ii)  $\{\alpha_i \mid i \in \bar{I}_{u_1} \cap \dots \cap \bar{I}_{u_k}\}$  if  $\beta(u_i) < 1$  for some  $i$ .

In addition, Theorem 4.3 in §4 is applicable to determine the isomorphism type of the identity component of the group  $C_H$ .  $\square$

### 3 Deficiency functions and their properties

According to (1.1) a centralizer  $C_{\exp(u)}$  with  $u \in \Delta$  admits the presentation

$$(3.1) \quad C_{\exp(u)} \cong (G_1 \times \dots \times G_k \times C_{\exp(u)}^{\text{Rad}})/H.$$

Moreover, its local type  $\pi : G_1 \times \cdots \times G_k \times C_{\exp(u)}^{Rad} \rightarrow C_{\exp(u)}$  can be read from the expression of  $u$  in (2.7) by Theorem 2.8. To complete this work it remains for us to decide in term of  $u$  the finite subgroup  $H \subseteq \mathcal{Z}(G_1) \times \cdots \times \mathcal{Z}(G_k) \times C_{\exp(u)}^{Rad}$  appearing as the denominator in (3.1). The main idea to do so is to introduce *the reduced weight system* of  $C_{\exp(u)}$ , as well as two *deficiency functions* on it, which play the role to clarify the difference between the group  $C_{\exp(u)}$  and its local type  $G_1 \times \cdots \times G_k \times C_{\exp(u)}^{Rad}$ .

### 3.1 Deficiency functions for semisimple Lie groups

We begin by introducing the reduced weight system and the deficiency functions for semisimple Lie groups, and demonstrate their use in specifying the isomorphism type of such a group.

Assume that  $G$  is a semisimple Lie group with local type  $\pi : G_1 \times \cdots \times G_k \rightarrow G$ . Fix a maximal torus  $T_i$  on each  $G_i$  and take  $T = \pi(T_1 \times \cdots \times T_k)$  as the fixed maximal torus on  $G$ . Then  $L(T) = L(T_1) \oplus \cdots \oplus L(T_k)$  and the tangent map of  $\pi$  at the group unit induces a partition

$$(3.3) \quad \Omega_G = \Omega_{G_1} \sqcup \cdots \sqcup \Omega_{G_k}$$

where  $\Omega_G$  (resp.  $\Omega_{G_i}$ ) is the set of fundamental dominant weights of  $G$  (resp. of  $G_i$ ) with respect to a fixed regular point  $(x_1, \dots, x_k) \in L(T)$ ,  $x_i \in L(T_i)$ .

**Definition 3.1.** With respect to the partition (3.3) *the reduced weight system* of the semisimple group  $G$  is the subset of its weight lattice  $\Lambda_G$ :

$$\Pi_G^0 = \{\theta_1 \oplus \cdots \oplus \theta_k \in \Lambda_G \mid \theta_i \in \Pi_{G_i} \sqcup \{0\}\}.$$

where  $\Pi_{G_i}$  is the set of minimal weights of the simple group  $G_i$  (see Table 2).

The integer valued function  $\delta_G : \Pi_G^0 \rightarrow \mathbb{Z}$  defined by

$$\delta_G(\theta) = \text{the order of the element } \exp(\theta) \text{ in the group } \mathcal{Z}(G), \theta \in \Pi_G^0,$$

is called the *deficiency function* of  $G$ .  $\square$

Let  $\Lambda_G^e = \exp^{-1}(e)$  be the *unit lattice* of  $G$ . In the Euclidean space  $L(T)$  one has three lattices  $\Lambda_G^r$ ,  $\Lambda_G^e$  and  $\Lambda_G$  that are subject to the relations

$$(3.4) \quad \Lambda_G^r \subseteq \Lambda_G^e \subseteq \Lambda_G.$$

Immediate, but useful properties of the function  $\delta_G$  are

**Corollary 3.2.** *Let  $G$  be a semisimple Lie group with center  $\mathcal{Z}(G)$ . Then the exponential map  $\exp : L(T) \rightarrow T$  satisfies  $\exp(\Pi_G^0) = \mathcal{Z}(G)$ . Moreover,*

i) *the value  $\delta_G(\theta)$  is the least positive multiple so that  $\delta_G(\theta) \cdot \theta \in \Lambda_G^e$ ;*

ii) *if  $G = G_1 \times G_2$ ,  $\delta_{G_1 \times G_2}(\theta_1 \oplus \theta_2) = \text{l.c.m. } \{\delta_{G_1}(\theta_1), \delta_{G_2}(\theta_2)\}$ ,*

*where  $\theta_i \in \Pi_{G_i}^0$ ,  $i = 1, 2$ , and where l.c.m. means the least common multiple of the indicated set of integers.*

**Proof.** Property i) comes from the fact that the exponential map  $\exp$  induces an one to one correspondences  $\Lambda_G / \Lambda_G^e \cong \mathcal{Z}(G)$ .

The item ii), together with the relation  $\exp(\Pi_G^0) = \mathcal{Z}(G)$ , follows from  $\exp(\Pi_{G_i} \sqcup \{0\}) = \mathcal{Z}(G_i)$  by (2.6), and  $\mathcal{Z}(G_1 \times G_2) = \mathcal{Z}(G_1) \times \mathcal{Z}(G_2)$ .  $\square$

**Example 3.3.** Let  $G$  be an 1-connected semisimple Lie group with reduced weight system  $\Pi_G^0$ . Properties i) and ii) of Corollary 3.2 is sufficient to evaluate the function  $\delta_G : \Pi_G^0 \rightarrow \mathbb{Z}$ .

a) If  $G$  is simple with Cartan matrix  $A$ , then the fundamental dominant weights  $\omega_1, \dots, \omega_n$  can be expressed by the simple roots as (by (2.5))

$$(3.5) \quad \omega_i = r_{i,1}\alpha_1 + \dots + r_{i,n}\alpha_n \text{ with } r_{i,k} \in \mathbb{Q}, A^{-1} = (r_{i,j})_{n \times n}.$$

Since  $\Lambda_G^r = \Lambda_G^e$  for the 1-connected Lie group  $G$ , the value  $\delta_G(\omega_i)$  with  $\omega_i \in \Pi_G$  is the least positive integer so that  $\delta_G(\omega_i) \cdot r_{i,k} \in \mathbb{Z}$  for all  $1 \leq k \leq n$ .

For all 1-connected simple Lie groups the expressions (3.5) can be found in [12, p.69], from which we can read off the function  $\delta_G : \Pi_G^0 \rightarrow \mathbb{Z}$  and tabulate its values in the third column of the table below:

$G$	$\Pi_G^0 = \Pi_G \amalg \{0\}$	$\delta_G(\theta), \theta \in \Lambda_G^0$
$SU(n+1)$	$\{\omega_1, \dots, \omega_n\} \amalg \{0\}$	$\{\frac{n+1}{(n+1,k)}\}_{1 \leq k \leq n} \amalg \{1\}$
$Sp(n)$	$\{\omega_n\} \amalg \{0\}$	$\{2\} \amalg \{1\}$
$Spin(2n+1)$	$\{\omega_1\} \amalg \{0\}$	$\{2\} \amalg \{1\}$
$Spin(4n)$	$\{\omega_1, \omega_{2n-1}, \omega_{2n}\} \amalg \{0\}$	$\{2, 2, 2\} \amalg \{1\}$
$Spin(4n+2)$	$\{\omega_1, \omega_{2n-1}, \omega_{2n}\} \amalg \{0\}$	$\{2, 4, 4\} \amalg \{1\}$
$E_6$	$\{\omega_1, \omega_6\} \amalg \{0\}$	$\{3, 3\} \amalg \{1\}$
$E_7$	$\{\omega_7\} \amalg \{0\}$	$\{2\} \amalg \{1\}$
$G_2, F_4, E_8$	$\{0\}$	$\{1\}$

Table 3. The deficiency function  $\delta_G : \Pi_G^0 \rightarrow \mathbb{Z}$  of 1-connected simple Lie groups.

b) If  $G$  is 1-connected with local type  $G = G_1 \times \dots \times G_k$ , by ii) of Corollary 3.2 the function  $\delta_G : \Pi_G^0 \rightarrow \mathbb{Z}$  is evaluated by

$$\delta_G(\theta_1 \oplus \dots \oplus \theta_k) = \text{l.c.m. } \{\delta_{G_1}(\theta_1), \dots, \delta_{G_k}(\theta_k)\},$$

where the values  $\delta_{G_j}(\theta_i)$  with  $\theta_i \in \Pi_{G_i}^0$  are given in Table 3.  $\square$

In general, assume that  $G$  is semisimple with local type

$$\pi : G^s = G_1 \times \dots \times G_k \rightarrow G.$$

The tangent map of  $\pi$  at the group unit of  $\tilde{T} = T_1 \times \dots \times T_k$  induces the canonical identifications  $L(\tilde{T}) = L(T)$ ,  $\Pi_G^0 = \Pi_{G^s}^0$ . It is in this sense that the reduced weight system  $\Pi_G^0$  of  $G$  possesses two deficiency functions:

$$\delta_G, \tilde{\delta}_G : \Pi_G^0 \rightarrow \mathbb{Z},$$

where  $\tilde{\delta}_G =: \delta_{G^s}$ . The next result tells that, comparison between these two functions enables one to specify  $\ker \pi$ , which determines the isomorphism type of  $G$ . For a finite group  $H$  write  $H^+$  for the set of all nontrivial elements in  $H$ .

**Lemma 3.4.** *Let  $G$  be semisimple with local type  $\pi : G^s \rightarrow G$ , and let  $\widetilde{\exp} : L(\tilde{T}) \rightarrow \tilde{T}$  be the exponential map of the maximal torus  $\tilde{T}$  on  $G^s$ . Then*

$$\ker \pi^+ = \{\widetilde{\exp}(\theta) \in G^s \mid \widetilde{\delta}_G(\theta) > \delta_G(\theta) = 1, \theta \in \Pi_G^0\},$$

where the order of an element  $\widetilde{\exp}(\theta) \in \ker \pi^+$  is  $\widetilde{\delta}_G(\theta)$ ,  $\theta \in \Pi_G^0$ .

**Proof.** Since  $\ker \pi \subseteq \mathcal{Z}(G^s) = \widetilde{\exp}(\Pi_{G^s}^0)$  by Corollary 3.2, any element of  $\ker \pi^+$  has the form  $\widetilde{\exp}(\theta)$  for some  $\theta \in \Pi_G^0$ .

On the other hand, for an element  $\theta \in \Pi_G^0$  the statements  $\widetilde{\delta}_G(\theta) > 1$  and  $\delta_G(\theta) = 1$  are clearly equivalent to  $\widetilde{\exp}(\theta) \in \ker \pi^+$ .  $\square$

### 3.2 Deficiency functions for connected Lie groups

We need to extend the deficiency functions to all compact connected Lie groups, so that an analogue of Lemma 3.4 holds for such a group. Assume therefore that  $K$  is a compact connected Lie group with local type

$$\pi : K^s \times K^{Rad} \rightarrow K, K^s = G_1 \times \cdots \times G_k$$

Taking a maximal torus  $T_i$  on each factor group  $G_i$  the quotient homomorphism  $\pi$  then carries the maximal torus  $\widetilde{T} = T_1 \times \cdots \times T_k \times K^{Rad}$  on  $K^s \times K^{Rad}$  onto the maximal torus  $T = \pi(\widetilde{T})$  on  $K$ . Moreover, the tangent map of  $\pi$  at the group unit induces an identification

$$L(\widetilde{T}) = L(T) = L(T_1) \oplus \cdots \oplus L(T_k) \oplus L(K^{Rad})$$

where  $L(K^{Rad})$  is the Lie algebra of the radical part  $K^{Rad}$ . Let  $\Pi_{G_1 \times \cdots \times G_k}^0 \subset L(T)$  be the reduced weight system of the semisimple part  $G_1 \times \cdots \times G_k$ .

The unit lattice  $\Lambda_{K^{Rad}}^e \subset L(K^{Rad})$  of the radical part  $K^{Rad}$  is obviously a subset of  $\Lambda_K^e$ . Let  $\Lambda_{K^{Rad}}^e(\mathbb{Q})$  be the vector space spanned by the elements in  $\Lambda_{K^{Rad}}^e$  over the rationals  $\mathbb{Q}$ , regarded as a subset of  $L(T)$ .

**Definition 3.5.** The *reduced weight lattice* of  $K$  is the subset of  $L(T)$

$$\Pi_K^0 = \{\omega \oplus \gamma \in L(T) \mid \omega \in \Pi_{G_1 \times \cdots \times G_k}^0, \gamma \in \Lambda_{K^{Rad}}^e(\mathbb{Q})\}.$$

The *deficiency function* of  $K$  is the integer valued map  $\delta_K : \Pi_K^0 \rightarrow \mathbb{Z}$  defined by

$$\delta_K(\omega \oplus \gamma) = \text{the least multiple so that } \delta_K(\omega \oplus \gamma) \cdot (\omega \oplus \gamma) \in \Lambda_K^e. \square$$

Again, the tangent map of  $\pi$  at the group unit induces a canonical identification  $\Pi_{K^s \times K^{Rad}}^0 = \Pi_K^0$  and therefore, in analogue to the semisimple cases, the set  $\Pi_K^0$  possesses two deficiency functions:

$$\widetilde{\delta}_K, \delta_K : \Pi_K^0 \rightarrow \mathbb{Z}, \widetilde{\delta}_K =: \delta_{K^s \times K^{Rad}}.$$

Clearly,  $\widetilde{\delta}_K : \Pi_K^0 \rightarrow \mathbb{Z}$  depends only on the local type of  $K$  in the sense that

$$\widetilde{\delta}_K(\theta_1 \oplus \cdots \oplus \theta_k \oplus \gamma) = \text{l.c.m. } \{\delta_{G_1}(\theta_1), \dots, \delta_{G_k}(\theta_k), \delta_{K^{Rad}}(\gamma)\},$$

where  $\theta_i \in \Pi_{G_i}^0$ ,  $\gamma \in \Lambda_{K^{Rad}}^e(\mathbb{Q})$ . The next result generalizes Lemma 3.4 from the semisimple Lie groups to all compact connected ones.

**Lemma 3.6.** *If  $K$  is compact connected Lie group with local type  $\pi : K^s \times K^{Rad} \rightarrow K$ . Then*

$$\ker \pi^+ = \{\widetilde{\exp}(\theta) \in K^s \times K^{Rad} \mid \widetilde{\delta}_K(\theta) > \delta_K(\theta) = 1, \theta \in \Pi_K^0\},$$

where the order of an element  $\widetilde{\exp}(\theta) \in \ker \pi^+$  is  $\widetilde{\delta}_K(\theta)$ ,  $\theta \in \Pi_K^0$ .  $\square$

### 3.3 The isomorphism type of a subgroup

Let  $K$  be a compact, connected subgroup of a semisimple Lie group  $G$  with inclusion  $h : K \rightarrow G$  and local type  $\pi : K^s \times K^{Rad} \rightarrow K$ . Assume that  $h$  carries a maximal torus  $T'$  of  $K$  into that  $T$  of  $G$ , and let  $h_* : L(T') \rightarrow L(T)$  be the tangent map of  $h$  at the group unit.

Since  $h$  is monomorphic, we have  $h_*^{-1}(\Lambda_G^e) = \Lambda_K^e$ . It follows that the condition  $\delta_K(\theta) = 1$ ,  $\theta \in \Pi_K^0$ , is equivalent to  $h_*\theta \in \Lambda_G^e$ . Therefore, one gets from Theorem 3.6 that

**Theorem 3.7.** *Let  $K$  be a compact, connected subgroup of a semisimple Lie group  $G$  with inclusion  $h : K \rightarrow G$  and local type  $\pi : K^s \times K^{Rad} \rightarrow K$ . Then*

$$(3.6) \quad \ker \pi^+ = \{\widetilde{\exp}(\theta) \in K^s \times K^{Rad} \mid \widetilde{\delta}_K(\theta) > 1, h_*(\theta) \in \Lambda_G^e, \theta \in \Pi_K^0\}. \square$$

## 4 The isomorphism types of centralizers in an 1-connected Lie group

To avoid case by case discussion we assume in this section that  $G$  is an 1-connected simple Lie group. Summarizing results in sections §2 and §3, our main result is presented in Theorem 4.3, which gives an explicit procedure for calculating the isomorphism type of a centralizer  $C_{\exp(u)}$ .

As applications we determine in §4.2 and §4.3 the isomorphism type of those centralizers  $C_{\exp(u)}$  with  $u \in \Delta$  a multiple of a fundamental dominant weight.

### 4.1 A procedure for calculating a centralizer $C_x$

For a group element  $x = \exp(u) \in G$  with  $u \in \Delta$  given as that in (2.7), Theorem 2.8 specifies the local type  $\pi$  of the centralizer  $C_x$ , hence the deficiency function  $\widetilde{\delta}_{C_x} : \Pi_{C_x}^0 \rightarrow \mathbb{Z}$ , see discussion in Section §3.2. In order to apply the formula (3.6) to compute the group  $\ker \pi$  we need to know the expressions of  $h_*(\theta)$ ,  $\theta \in \Pi_{C_x}^0$ , in term of simple roots (or equivalently, the fundamental dominant weights) of the group  $G$ , where  $h : C_{\exp(u)} \rightarrow G$  is the inclusion.

Let  $A$  (resp.  $\widetilde{A}$ ) be the Cartan matrix (resp. the extended Cartan matrix) of the group  $G$  with respect to the system  $\{\alpha_1, \dots, \alpha_n\}$  of simple roots (resp. the extended system  $\{\alpha_1, \dots, \alpha_n, -\beta\}$  of simple roots). As in (2.7) for a vector  $u \in \Delta$  assume that  $I_u = \{k_1, \dots, k_r\}$  and let  $\widetilde{I}_u = \{j_1, \dots, j_{n-r}\}$  be the complement of the ordered sequence  $I_u$  in  $\{1, \dots, n\}$ . Let  $A_u$  (resp.  $\widetilde{A}_u$ ) be the matrix obtained from the matrix  $A$  (resp.  $\widetilde{A}$ ) by deleting all the  $i^{th}$  columns and rows with  $i \in I_u$ .

Assume that the set  $\Omega_{C_{\exp(u)}^s}$  of fundamental dominant weights of the semisimple part  $C_{\exp(u)}^s$  is  $\{\omega'_1, \dots, \omega'_s\}$ , where

$$s = n - r \text{ if } \beta(u) < 1, \text{ and } n - r + 1 \text{ if } \beta(u) = 1$$

As direct consequences of Theorem 2.8 we have

**Lemma 4.1.** *The tangent map  $h_*$  of  $h$  at the group unit satisfies that*  
*i) if  $\beta(u) < 1$  then*

$$\begin{pmatrix} h_*(\omega'_1) \\ \vdots \\ h_*(\omega'_{n-r}) \end{pmatrix} = A_u^{-1} \begin{pmatrix} \alpha_{j_1} \\ \vdots \\ \alpha_{j_{n-r}} \end{pmatrix}$$

and

$$i_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{a_1\omega_{k_1} + \cdots + a_r\omega_{k_r} \mid a_i \in \mathbb{Q}\};$$

ii) if  $\beta(u) = 1$  then

$$\begin{pmatrix} h_*(\omega'_1) \\ \vdots \\ h_*(\omega'_{n-r}) \\ h_*(\omega'_{n-r+1}) \end{pmatrix} = \tilde{A}_u^{-1} \begin{pmatrix} \alpha_{j_1} \\ \vdots \\ \alpha_{j_{n-r}} \\ -\beta \end{pmatrix}$$

and

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{a_1\omega_{k_1} + \cdots + a_r\omega_{k_r} \mid \Sigma a_i \beta^*(\omega_{k_i}) = 0, a_i \in \mathbb{Q}\}.$$

**Proof.** The formula of  $h_*(\omega'_i)$  comes from (2.5). The expressions of  $h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q}))$  follows from the relation  $C_{\exp(u)}^{Rad} = T_u$  or  $T_u^\beta$  by Theorem 2.8, as well as the definition of the groups  $T_u$  and  $T_u^\beta$  in §2.4.  $\square$

In general a centralizer  $C_{\exp(u)}$  may not be semisimple. As a result its extended weight system  $\Pi_{C_{\exp(u)}}^0$  might contain the infinite factor  $\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q}) \subset \Pi_{C_{\exp(u)}^x}^0$ , see Definition 3.5. This raises the question whether the deficiency function  $\tilde{\delta}_{C_x} : \Pi_{C_x}^0 \rightarrow \mathbb{Z}$  can be effectively calculated. The next result allows us to reduce the determination of  $\ker \pi^+$  by dealing with the finite set

$$(4.1) \quad H_u = \{\theta \in \Pi_{C_{\exp(u)}}^0 \mid \delta_{C_{\exp(u)}}^s(\theta) > 1, h_*(\theta) \in h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) \bmod \Lambda_G^r\},$$

which, in practice, can be easily decided from the concrete expressions of  $h_*(\theta)$  (with  $\theta \in \Pi_{C_{\exp(u)}}^0$ ) and  $h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q}))$  by Lemma 4.1. Note that with the assumption that  $G$  is 1-connected, one has  $\Lambda_G^r = \Lambda_G^e$ .

**Theorem 4.2.** *Let  $\pi$  be the local type of the centralizer  $C_{\exp(u)}$ . Then*

$$(4.2) \quad \ker \pi^+ = \{\widetilde{\exp}(\theta - \gamma_\theta) \mid \theta \in H_u\},$$

where  $\gamma_\theta \in \Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})$  is an arbitrary element satisfying

$$h_*(\theta) \equiv h_*(\gamma_\theta) \bmod \Lambda_G^r.$$

**Proof.** The exponential map  $\widetilde{\exp}$  of the local type  $C_{\exp(u)}^s \times C_{\exp(u)}^{Rad}$  of  $C_{\exp(u)}$  will be written as  $\exp_1 \times \exp_2$ , where  $\exp_1$  and  $\exp_2$  are the exponential maps of the factors  $C_{\exp(u)}^s$  and  $C_{\exp(u)}^{Rad}$ , respectively. Let

$$\pi_2 : C_{\exp(u)}^{Rad} \rightarrow C_{\exp(u)}$$

be the restriction of  $\pi$  on the second factor  $C_{\exp(u)}^{Rad}$ . By the definition of the radical part  $K^{Rad}$  of a Lie group  $K$  (see §1) the map  $\pi_2$  (hence the composition  $h \circ \pi_2$ ) is injective.

For a  $\theta \in H_u$  let  $\gamma_\theta \in \Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})$  be a vector with  $h_*(\theta) \equiv h_*(\gamma_\theta) \bmod \Lambda_G^r$ . We get from

$$\widetilde{\delta}_{C_{\exp(u)}}(\theta - \gamma_\theta) = \text{l.c.m.} \{ \delta_{C_{\exp(u)}^s}(\theta), \delta_{C_{\exp(u)}^{Rad}}(-\gamma_\theta) \} \geq \delta_{C_{\exp(u)}^s}(\theta) > 1$$

and

$$h_*(\theta - \gamma_\theta) \in \Lambda_G^r \text{ (since } h_*(\theta) \equiv h_*(\gamma_\theta) \bmod \Lambda_G^r \text{)}$$

that  $\widetilde{\exp}(\theta - \gamma_\theta) \in \ker \pi^+$  by Theorem 3.7. Furthermore, the element  $\widetilde{\exp}(\theta - \gamma_\theta)$  is independent of the choice of  $\gamma_\theta$  since if  $\gamma'_\theta \in \Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})$  is a second one with  $h_*(\theta) \equiv h_*(\gamma'_\theta) \bmod \Lambda_G^r$ , then the relation

$$h_*(\gamma_\theta) = h_*(\gamma'_\theta) \bmod \Lambda_G^r$$

and the injectivity of  $h \circ \pi_2$  imply that

$$\exp_2(\gamma_\theta) = \exp_2(\gamma'_\theta) \text{ (in } C_{\exp(u)}^{Rad} \text{)}.$$

Consequently,

$$\begin{aligned} \widetilde{\exp}(\theta - \gamma_\theta) &= \exp_1(\theta) \times \exp_2(-\gamma_\theta) \\ &= \exp_1(\theta) \times \exp_2(-\gamma'_\theta) = \widetilde{\exp}(\theta - \gamma'_\theta) \text{ (in } C_{\exp(u)}^s \times C_{\exp(u)}^{Rad} \text{)}. \end{aligned}$$

Conversely, for any pair  $(\theta, \gamma) \in \Pi_{C_{\exp(u)}^s}^0 \times \Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q}) (= \Pi_{C_{\exp(u)}}^0)$  with  $\widetilde{\exp}(\theta - \gamma) \in \ker \pi^+$ , we get from  $h_*(\theta - \gamma) \in \Lambda_G^r$  that  $h_*(\theta) \equiv h_*(\gamma) \bmod \Lambda_G^r$ , and from

$$\widetilde{\delta}_{C_{\exp(u)}}(\theta - \gamma) = \text{l.c.m.} \{ \delta_{C_{\exp(u)}^s}(\theta), \delta_{C_{\exp(u)}^{Rad}}(-\gamma) \} > 1$$

and the injectivity of  $h \circ \pi_2$  that  $\delta_{C_{\exp(u)}^s}(\theta) > 1$ . This completes the proof.  $\square$

Summarizing the results in Theorems 2.8, 4.2 and Lemma 4.1, we obtain the next explicit procedure for calculating  $C_{\exp(u)}$  in term of  $u \in \Delta$ .

**Theorem 4.3.** *Let  $G$  be a 1-connected simple Lie group with fundamental Weyl cell  $\Delta$ . For an vector  $u \in \Delta$  with  $u \notin \Omega_G$  the isomorphism type of  $C_{\exp(u)}$  can be obtained by the procedure below:*

**Step 1.** *Apply Theorem 2.8 to get the local type of  $C_{\exp(u)}$  in the form  $G_1 \times \cdots \times G_k \times C_x^{Rad}$  with each  $G_i$  an 1-connected and semisimple Lie group. Accordingly, write the reduced weight system of the semisimple part  $C_{\exp(u)}^s$  as*

$$\Pi_{C_{\exp(u)}^s}^0 = \{ \theta_1 \oplus \cdots \oplus \theta_k \mid \theta_i \in \Pi_{G_i} \sqcup \{0\} \}.$$

**Step 2.** *Apply Lemma 4.1 to get the expressions of the vectors  $h_*(\theta)$  (with  $\theta \in \Pi_{G_i}^0$ ) and the subspace  $h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q}))$  in  $\Lambda_G(\mathbb{Q})$ . Accordingly, specify the finite set*

$$H_u = \{\theta \in \Pi_{C_{\exp(u)}^s}^0 \mid \delta_{C_{\exp(u)}^s}(\theta) > 1, h_*(\theta) \in h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) \bmod \Lambda_G^r\}.$$

**Step 3.** The group  $C_{\exp(u)}$  is isomorphic to  $G_1 \times \cdots \times G_k \times C_x^{Rad} / \ker \pi$  with

$$\ker \pi^+ = \{\widetilde{\exp}(\theta - \gamma_\theta) \in C_x^s \times C_x^{Rad} \mid \theta \in H_u\},$$

where  $\gamma_\theta \in \Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})$  an element satisfying  $h_*(\theta) \equiv h_*(\gamma_\theta) \bmod \Lambda_G^r$ .  $\square$

Finally, concerning the structure of  $\ker \pi$  as a group, the next observation from the relation (4.2) will be repeatedly used in the forthcoming calculation:

(4.3) if the set  $H_u$  contains  $p - 1$  elements and if  $\theta_0 \in H_u$  is an element with  $\delta_{C_{\exp(u)}^s}(\theta_0) = p$ , then  $\ker \pi$  is the cyclic group  $\mathbb{Z}_p$  of order  $p$  generated by the element  $\widetilde{\exp}(\theta_0 - \gamma_{\theta_0}) \in G_1 \times \cdots \times G_k \times C_x^{Rad}$ .  $\square$

## 4.2 The maximal subgroups of maximal rank in a Lie group

Let  $G$  be an 1-connected exceptional Lie group with rank  $n$ . From the expression of the maximal short root  $\beta$  given in [12, p.66], the set  $\mathcal{F}_G$  (see §2.4)) is determined and presented in the second column of Table 4 below. Applying Theorem 2.8 one obtains the local types of the centralizers  $C_{\exp(u)}$ ,  $u \in \mathcal{F}_G$ , that are presented in the third column of the table.

In view of the explicit presentation of  $\mathcal{F}_G$  in the second column of the table we note that elements in  $\mathcal{F}_G$  are of the form  $u_i = \frac{\omega_i}{p_i}$  with  $p_i > 0$  an integer,  $1 \leq i \leq n$ . According to Borel and Siebenthal [4, Theorem 6] we have

(4.4) the centralizer  $C_{\exp(u_i)}$  with  $u_i = \frac{\omega_i}{p_i} \in \mathcal{F}_G$  is a maximal subgroup of maximal rank of  $G$  if and only if  $p_i$  is a prime.

Carrying on discussion in Corollary 2.9 and Remark 2.10 we determine the isomorphism types of the centralizer  $C_{\exp(u_i)}$  for all  $u_i \in \mathcal{F}_G$ . In order to make the generators of  $\ker \pi$  explicit, for a product  $K = K_1 \times K_2$  of two groups we write  $\exp_1 \times \exp_2$  instead of  $\exp$ , where  $\exp$  (resp.  $\exp_i$ ,  $i = 1, 2$ ) is the exponential map of the group  $K$  (resp. of  $K_i$ ,  $i = 1, 2$ ).

**Theorem 4.4.** *Let  $G$  be an 1-connected exceptional Lie group. The isomorphism types of the centralizers  $C_{\exp(u)}$  with  $u \in \mathcal{F}_G$  are given by the third and fourth columns of the table below, in which those are of maximal in  $G$  are specified by (4.4) (compare with the table in [4, §7]).*



$G$	$u \in F_G$	Local type of $C_{\exp(u)}$	$\ker \pi$ (generator)
$G_2$	$\frac{\omega_1}{2}$	$SU(2) \times SU(2)$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2))$
	$\frac{\omega_2}{2}$	$SU(3)$	$Z_3(\exp_1(\omega_1^1))$
$F_4$	$\frac{\omega_1}{2}$	$Spin(9)$	$\{0\}$
	$\frac{\omega_2}{2}$	$SU(2) \times SU(4)$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2))$
	$\frac{\omega_3}{2}$	$SU(3) \times SU(3)$	$Z_3(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2))$
	$\frac{\omega_4}{2}$	$Sp(3) \times SU(2)$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2))$
$E_6$	$\frac{\omega_1}{2}, \frac{\omega_6}{2}$	$Spin(10) \times S^1$	$Z_4(\exp_1(\omega_1^1) \times \exp_2(-\frac{9}{4}\omega_{1(6)}))$
	$\frac{\omega_2}{2}, \frac{\omega_3}{2}, \frac{\omega_5}{2}$	$SU(2) \times SU(6)$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2))$
	$\frac{\omega_4}{2}$	$SU(3) \times SU(3) \times SU(3)$	$Z_3(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2) \times \exp_3(\omega_3^3))$
$E_7$	$\frac{\omega_1}{2}, \frac{\omega_6}{2}$	$Spin(12) \times SU(2)$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2))$
	$\frac{\omega_2}{2}$	$SU(8)$	$Z_2(\exp(\omega_4^1))$
	$\frac{\omega_3}{2}, \frac{\omega_5}{3}$	$SU(3) \times SU(6)$	$Z_3(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2))$
	$\frac{\omega_4}{2}$	$SU(2) \times SU(4) \times SU(4)$	$Z_4(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(\omega_3^3))$
	$\frac{\omega_7}{2}$	$E_6 \times S^1$	$Z_3(\exp_1(\omega_1^1) \times \exp_2(-\frac{4}{3}\omega_7))$
$E_8$	$\frac{\omega_1}{2}$	$Spin(16)$	$Z_2(\exp(\omega_7^1))$
	$\frac{\omega_2}{2}$	$SU(9)$	$Z_3(\exp(\omega_3^1))$
	$\frac{\omega_3}{2}$	$SU(8) \times SU(2)$	$Z_4(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2))$
	$\frac{\omega_4}{2}$	$SU(2) \times SU(3) \times SU(6)$	$Z_6(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(\omega_5^3))$
	$\frac{\omega_5}{2}$	$SU(5) \times SU(5)$	$Z_5(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2))$
	$\frac{\omega_6}{2}$	$Spin(10) \times SU(4)$	$Z_4(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2))$
	$\frac{\omega_7}{2}$	$E_6 \times SU(3)$	$Z_3(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2))$
	$\frac{\omega_8}{2}$	$E_7 \times SU(2)$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2))$

Table 4. The maximal subgroups of the maximal rank of exceptional Lie groups

**Proof.** It suffices to establish the results in fifth columns of the table. These will be done by applying the steps 2 and 3 entailed in Theorem 4.3. The calculations will be divided into five cases in accordance to  $G = G_2, F_4, E_6, E_7$  and  $E_8$ .

If  $G = E_6$ ,  $u = \frac{\omega_1}{2}, \frac{\omega_6}{2}$  (resp.  $G = E_7$ ,  $u = \frac{\omega_7}{2}$ ), the centralizer  $C_{\exp(u)}$  are parabolic. The proofs of these cases will be postponed to the next section.

In the remaining cases the centralizers  $C_{\exp(u)}$  are always semisimple. Therefore, the constraint  $h_*(\theta) \in h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) \bmod \Lambda_G^r$  on the set  $H_u$  (see (4.1)) is equivalent to  $h_*(\theta) \equiv 0 \bmod \Lambda_G^r$ .

It is more convenient for us to express  $h_*(\theta)$  with  $\theta \in \Pi_{C_{\exp(u)}}$  in term of the weights of  $G$  (instead the simple roots). For the transitions from weights to roots we refer to the table in [12, p.69].

**Case 1.**  $G = G_2$ .

i) If  $u = \frac{\omega_1}{2}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(2)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1\} \amalg \{\omega_1^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(2)}$  by fundamental dominant weights  $\omega_1, \omega_2$  of  $G_2$ :

$$h_*(\omega_1^1) = -\frac{\omega_1}{2}; \quad h_*(\omega_1^2) = \omega_2 - \frac{3}{2}\omega_1.$$

It follows that the set  $H_u$  consists of the single element  $\omega_1^1 \oplus \omega_1^2$  whose deficiency in the group  $SU(2) \times SU(2)$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_1^2)$ .

ii) If  $u = \frac{\omega_2}{3}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(3)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(3)}$  by the fundamental dominant weights  $\omega_1, \omega_2$  of  $G_2$ :

$$h_*(\omega_1^1) = -\omega_2; \quad h_*(\omega_2^1) = \omega_1 - 2\omega_2.$$

It follows that the set  $H_u$  consists of two elements  $\omega_1^1$  and  $\omega_2^1$  whose deficiencies in the group  $SU(3)$  are both 3.

Consequently,  $\ker \pi = \mathbb{Z}_3$  with generator  $\exp_1(\omega_1^1)$  by (4.3).

**Case 2.**  $G = F_4$

i) If  $u = \frac{\omega_1}{2}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $Spin(9)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{Spin(9)} = \{\omega_1^1\}$  by fundamental dominant weights of  $F_4$

$$h_*(\omega_1^1) = -\frac{\omega_1}{2}.$$

It follows that  $H_u = \emptyset$ . Consequently,  $\ker \pi = \{0\}$ .

ii) If  $u = \frac{\omega_2}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(4)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(4)}$  by the fundamental dominant weights of  $F_4$

$$\begin{aligned} h_*(\omega_1^1) &= \omega_1 - \frac{1}{2}\omega_2 \\ h_*(\omega_1^2) &= \omega_3 - \frac{3}{4}\omega_2 \\ h_*(\omega_2^2) &= \omega_4 - \frac{1}{2}\omega_2 \\ h_*(\omega_3^2) &= -\frac{1}{4}\omega_2. \end{aligned}$$

It follows that the set  $H_u$  consists of the single element  $\omega_1^1 \oplus \omega_2^2$  whose deficiency in the group  $SU(2) \times SU(4)$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_2^2)$ .

iii) If  $u = \frac{\omega_3}{3}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(3) \times SU(3)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1\} \amalg \{\omega_1^2, \omega_2^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(3)} \sqcup \Pi_{SU(3)}$  by the fundamental dominant weights of  $F_4$ :

$$\begin{aligned} h_*(\omega_1^1) &= \omega_1 - \frac{2}{3}\omega_3 \\ h_*(\omega_2^1) &= \omega_2 - \frac{4}{3}\omega_3 \\ h_*(\omega_1^2) &= \omega_4 - \frac{2}{3}\omega_3 \\ h_*(\omega_2^2) &= -\frac{1}{3}\omega_3. \end{aligned}$$

It follows that the set  $H_u$  consists of the two elements  $\omega_1^1 \oplus \omega_2^2$  and  $\omega_2^1 \oplus \omega_1^2$  whose deficiencies in the group  $SU(3) \times SU(3)$  are both 3.

Consequently,  $\ker \pi = \mathbb{Z}_3$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_2^2)$  by (4.3).

iv) If  $u = \frac{\omega_4}{2}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $Sp(3) \times SU(2)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1\} \amalg \{\omega_1^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{Sp(3)} \sqcup \Pi_{SU(2)}$  by the fundamental dominant weights of  $F_4$ :

$$h_*(\omega_3^1) = \omega_3 - \frac{3}{2}\omega_4; \quad h_*(\omega_1^2) = -\frac{1}{2}\omega_4.$$

It follows that the set  $H_u$  consists of the single element  $\omega_3^1 \oplus \omega_1^2$  whose deficiency in the group  $Sp(3) \times SU(2)$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp_1(\omega_3^1) \times \exp_2(\omega_1^2)$ .

**Case 3.**  $G = E_6$ .

i)  $u = \frac{\omega_1}{2}$  (resp.  $\frac{\omega_6}{2}$ ). See in the proof of Theorem 4.6.

ii) If  $u = \frac{\omega_2}{2}$  (resp.  $\frac{\omega_3}{2}, \frac{\omega_5}{2}$ ), the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(6)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2, \omega_5^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(6)}$  by the weights of  $E_6$ :

$$\begin{aligned} h_*(\omega_1^1) &= -\frac{1}{2}\omega_2 \text{ (resp. } \omega_1 - \frac{1}{2}\omega_3; \omega_6 - \frac{1}{2}\omega_5) \\ h_*(\omega_1^2) &= \omega_1 - \frac{1}{2}\omega_2 \text{ (resp. } \omega_6 - \frac{1}{2}\omega_3; \omega_1 - \frac{1}{2}\omega_5) \\ h_*(\omega_2^2) &= \omega_3 - \omega_2 \text{ (resp. } \omega_5 - \omega_3; \omega_3 - \omega_5) \\ h_*(\omega_3^2) &= \omega_4 - \frac{3}{2}\omega_2 \text{ (resp. } \omega_4 - \frac{3}{2}\omega_3; \omega_4 - \frac{3}{2}\omega_5) \\ h_*(\omega_4^2) &= \omega_5 - \omega_2 \text{ (resp. } \omega_2 - \omega_3; \omega_2 - \omega_5) \\ h_*(\omega_5^2) &= \omega_6 - \frac{1}{2}\omega_2 \text{ (resp. } -\frac{1}{2}\omega_3; -\frac{1}{2}\omega_5) \end{aligned}$$

It follows that the set  $H_u$  consists of the single element  $\omega_1^1 \oplus \omega_3^2$  whose deficiency in the group  $SU(2) \times SU(6)$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_3^2)$ .

iii) If  $u = \frac{\omega_4}{3}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(3) \times SU(3) \times SU(3)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1\} \amalg \{\omega_1^2, \omega_2^2\} \amalg \{\omega_1^3, \omega_2^3\}$ . Applying Lemma 4.1, we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(3)} \sqcup \Pi_{SU(3)} \sqcup \Pi_{SU(3)}$  by the weights of  $E_6$ :

$$\begin{aligned} h_*(\omega_1^1) &= \omega_1 - \frac{1}{3}\omega_4 \\ h_*(\omega_2^1) &= \omega_3 - \frac{2}{3}\omega_4 \\ h_*(\omega_1^2) &= \omega_2 - \frac{2}{3}\omega_4 \\ h_*(\omega_2^2) &= -\frac{1}{3}\omega_4 \\ h_*(\omega_1^3) &= \omega_5 - \frac{2}{3}\omega_4 \\ h_*(\omega_2^3) &= \omega_6 - \frac{1}{3}\omega_4. \end{aligned}$$

It follows that the set  $H_u$  consists of the two elements  $\omega_2^1 \oplus \omega_1^2 \oplus \omega_1^3$  and  $\omega_1^1 \oplus \omega_2^2 \oplus \omega_2^3$  whose deficiency in the group  $SU(3) \times SU(3) \times SU(3)$  are both 3.

Consequently,  $\ker \pi = \mathbb{Z}_3$  with generator  $\exp_1(\omega_2^1) \times \exp_2(\omega_1^2) \times \exp_3(\omega_1^3)$  by (4.3).

**Case 4.**  $G = E_7$

i) If  $u = \frac{\omega_1}{2}$  (resp.  $\frac{\omega_6}{2}$ ), the local type of the centralizer  $C_{\exp(u)}$  is  $Spin(12) \times SU(2)$ . Accordingly, assume the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1\} \amalg \{\omega_1^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{Spin(12)} \sqcup \Pi_{SU(2)}$  by the weights of  $E_7$ :

$$\begin{aligned} h_*(\omega_1^1) &= \omega_7 - \frac{1}{2}\omega_1 \text{ (resp. } -\frac{1}{2}\omega_6) \\ h_*(\omega_5^1) &= \omega_3 - \frac{3}{2}\omega_1 \text{ (resp. } \omega_5 - \frac{3}{2}\omega_6) \\ h_*(\omega_6^1) &= \omega_2 - \omega_1 \text{ (resp. } \omega_2 - \omega_6) \\ h_*(\omega_1^2) &= -\frac{1}{2}\omega_1 \text{ (resp. } \omega_7 - \frac{1}{2}\omega_6). \end{aligned}$$

It follows that the set  $H_u$  consists of the single element  $\omega_5^1 \oplus \omega_1^2$  whose deficiency in the group  $Spin(12) \times SU(2)$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp_1(\omega_5^1) \times \exp_2(\omega_1^2)$ .

ii) If  $u = \frac{\omega_2}{2}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(8)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1, \omega_7^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(8)}$  by the weights of  $E_7$ :

$$\begin{aligned} h_*(\omega_1^1) &= -\frac{1}{2}\omega_2 \\ h_*(\omega_2^1) &= \omega_1 - \omega_2 \\ h_*(\omega_3^1) &= \omega_3 - \frac{3}{2}\omega_2 \\ h_*(\omega_4^1) &= \omega_4 - 2\omega_2 \\ h_*(\omega_5^1) &= \omega_5 - \frac{3}{2}\omega_2 \\ h_*(\omega_6^1) &= \omega_6 - \omega_2 \\ h_*(\omega_7^1) &= \omega_7 - \frac{1}{2}\omega_2 \end{aligned}$$

It follows that the set  $H_u$  consists of the single element  $\omega_4^1$  whose deficiency in the group  $SU(8)$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp(\omega_4^1)$ .

iii) If  $u = \frac{\omega_3}{3}$  (resp.  $\frac{\omega_5}{3}$ ), the local type of the centralizer  $C_{\exp(u)}$  is  $SU(3) \times SU(6)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2, \omega_5^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(3)} \sqcup \Pi_{SU(6)}$  by the weights of  $E_7$ :

$$\begin{aligned} h_*(\omega_1^1) &= -\frac{1}{3}\omega_3 \text{ (resp. } \omega_6 - \frac{2}{3}\omega_5) \\ h_*(\omega_2^1) &= \omega_1 - \frac{2}{3}\omega_3 \text{ (resp. } \omega_7 - \frac{1}{3}\omega_5) \\ h_*(\omega_1^2) &= \omega_2 - \frac{2}{3}\omega_3 \text{ (resp. } -\frac{1}{3}\omega_5) \\ h_*(\omega_2^2) &= \omega_4 - \frac{4}{3}\omega_3 \text{ (resp. } \omega_1 - \frac{2}{3}\omega_5) \\ h_*(\omega_3^2) &= \omega_5 - \omega_3 \text{ (resp. } \omega_3 - \omega_5) \\ h_*(\omega_4^2) &= \omega_6 - \frac{2}{3}\omega_3 \text{ (resp. } \omega_4 - \frac{4}{3}\omega_5) \\ h_*(\omega_5^2) &= \omega_7 - \frac{1}{3}\omega_3 \text{ (resp. } \omega_2 - \frac{2}{3}\omega_5). \end{aligned}$$

It follows that the set  $H_u$  consists of the two elements  $\omega_1^1 \oplus \omega_4^2$  and  $\omega_2^1 \oplus \omega_2^2$  whose deficiencies in the group  $SU(3) \times SU(6)$  is 3.

Consequently,  $\ker \pi = \mathbb{Z}_3$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_4^2)$  by (4.3).

iv) If  $u = \frac{\omega_4}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(4) \times SU(4)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2\} \amalg \{\omega_1^3, \omega_2^3, \omega_3^3\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(4)} \sqcup \Pi_{SU(4)}$  by the weights of  $E_7$ :

$$\begin{aligned} h_*(\omega_1^1) &= \omega_2 - \frac{1}{2}\omega_4 \\ h_*(\omega_1^2) &= -\frac{1}{4}\omega_4 \\ h_*(\omega_2^2) &= \omega_1 - \frac{1}{2}\omega_4 \\ h_*(\omega_3^2) &= \omega_3 - \frac{3}{4}\omega_4 \end{aligned}$$

$$\begin{aligned}
h_*(\omega_1^3) &= \omega_5 - \frac{3}{4}\omega_4 \\
h_*(\omega_2^3) &= \omega_6 - \frac{1}{2}\omega_4 \\
h_*(\omega_3^3) &= \omega_7 - \frac{1}{4}\omega_4.
\end{aligned}$$

It follows that the set  $H_u$  consists of the 3 elements

$$\omega_2^2 \oplus \omega_3^3, \omega_1^1 \oplus \omega_1^2 \oplus \omega_3^3, \omega_1^1 \oplus \omega_3^2 \oplus \omega_1^3$$

whose deficiencies in the group  $SU(2) \times SU(4) \times SU(4)$  are 2, 4, 4.

Consequently,  $\ker \pi = \mathbb{Z}_4$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(\omega_3^3)$  by (4.3).

vi)  $u = \frac{\omega_7}{2}$ . See in the proof of Theorem 4.6.

**Case 5.**  $G = E_8$

i) If  $u = \frac{\omega_1}{2}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $Spin(16)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1, \omega_7^1, \omega_8^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{Spin(16)} = \{\omega_1^1, \omega_7^1, \omega_8^1\}$  by the weights of  $E_8$

$$\begin{aligned}
h_*(\omega_1^1) &= -\frac{1}{2}\omega_1 \\
h_*(\omega_7^1) &= \omega_3 - 2\omega_1 \\
h_*(\omega_8^1) &= \omega_2 - \frac{3}{2}\omega_1
\end{aligned}$$

It follows that the set  $H_u$  consists of the single element  $\omega_7^1$  whose deficiency in the group  $Spin(16)$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp(\omega_7^1)$ .

ii) If  $u = \frac{\omega_2}{3}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(9)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1, \omega_7^1, \omega_8^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(9)}$  by the weights of  $E_8$ :

$$\begin{aligned}
h_*(\omega_1^1) &= \omega_1 - \frac{2}{3}\omega_2 \\
h_*(\omega_2^1) &= \omega_3 - \frac{4}{3}\omega_2 \\
h_*(\omega_3^1) &= \omega_4 - 2\omega_2 \\
h_*(\omega_4^1) &= \omega_5 - \frac{5}{3}\omega_2 \\
h_*(\omega_5^1) &= \omega_6 - \frac{4}{3}\omega_2 \\
h_*(\omega_6^1) &= \omega_7 - \omega_2 \\
h_*(\omega_7^1) &= \omega_8 - \frac{2}{3}\omega_2 \\
h_*(\omega_8^1) &= -\frac{1}{3}\omega_2.
\end{aligned}$$

It follows that the set  $H_u$  consists of the two elements  $\omega_3^1$  and  $\omega_6^1$  whose deficiencies in the group  $SU(9)$  are both 3.

Consequently,  $\ker \pi = \mathbb{Z}_3$  with generator  $\exp(\omega_3^1)$  by (4.3).

iii) If  $u = \frac{\omega_3}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(8) \times SU(2)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1, \omega_7^1\} \amalg \{\omega_1^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(8)} \sqcup \Pi_{SU(2)}$  by the weights of  $E_8$ :

$$\begin{aligned}
h_*(\omega_1^1) &= \omega_2 - \frac{3}{4}\omega_3 \\
h_*(\omega_2^1) &= \omega_4 - \frac{3}{2}\omega_3 \\
h_*(\omega_3^1) &= \omega_5 - \frac{5}{4}\omega_3 \\
h_*(\omega_4^1) &= \omega_6 - \omega_3 \\
h_*(\omega_5^1) &= \omega_7 - \frac{3}{4}\omega_3 \\
h_*(\omega_6^1) &= \omega_8 - \frac{1}{2}\omega_3 \\
h_*(\omega_7^1) &= -\frac{1}{4}\omega_3 \\
h_*(\omega_1^2) &= \omega_1 - \frac{1}{2}\omega_3
\end{aligned}$$

It follows that the set  $H_u$  consists of the 3 elements  $\omega_4^1, \omega_2^1 \oplus \omega_1^2, \omega_6^1 \oplus \omega_1^2$  in which  $\omega_2^1 \oplus \omega_1^2$  has deficiency 4 in the group  $SU(8) \times SU(2)$ .

Consequently,  $\ker \pi = \mathbb{Z}_4$  with generator  $\exp_1(\omega_2^1) \times \exp_2(\omega_1^2)$  by (4.3).

iv) If  $u = \frac{\omega_4}{6}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(3) \times SU(6)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2\} \amalg \{\omega_1^3, \omega_2^3, \omega_3^3, \omega_4^3, \omega_5^3\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(3)} \sqcup \Pi_{SU(6)}$  by the weights of  $E_8$ :

$$\begin{aligned}
h_*(\omega_1^1) &= \omega_2 - \frac{1}{2}\omega_4 \\
h_*(\omega_1^2) &= \omega_1 - \frac{1}{3}\omega_4 \\
h_*(\omega_2^2) &= \omega_3 - \frac{2}{3}\omega_4 \\
h_*(\omega_1^3) &= \omega_5 - \frac{5}{6}\omega_4 \\
h_*(\omega_2^3) &= \omega_6 - \frac{2}{3}\omega_4 \\
h_*(\omega_3^3) &= \omega_7 - \frac{1}{2}\omega_4 \\
h_*(\omega_4^3) &= \omega_8 - \frac{1}{3}\omega_4 \\
h_*(\omega_5^3) &= -\frac{1}{6}\omega_4
\end{aligned}$$

It follows that the set  $H_u$  consists of the 5 elements

$$\omega_1^1 \oplus \omega_1^2 \oplus \omega_5^3, \omega_1^1 \oplus \omega_3^3, \omega_1^2 \oplus \omega_2^3, \omega_2^2 \oplus \omega_4^3, \omega_1^3 \oplus \omega_5^3$$

in which  $\omega_1^1 \oplus \omega_1^2 \oplus \omega_5^3$  has deficiency 6 in the group  $SU(2) \times SU(3) \times SU(6)$ .

Consequently,  $\ker \pi = \mathbb{Z}_6$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(\omega_5^3)$ .

v) If  $u = \frac{\omega_5}{5}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(5) \times SU(5)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Omega = \Pi_{SU(5)} \sqcup \Pi_{SU(5)}$  by the weights of  $E_8$ :

$$\begin{aligned}
h_*(\omega_1^1) &= \omega_1 - \frac{2}{5}\omega_5 \\
h_*(\omega_2^1) &= \omega_3 - \frac{4}{5}\omega_5 \\
h_*(\omega_3^1) &= \omega_4 - \frac{6}{5}\omega_5 \\
h_*(\omega_4^1) &= \omega_2 - \frac{3}{5}\omega_5 \\
h_*(\omega_1^2) &= \omega_6 - \frac{4}{5}\omega_5 \\
h_*(\omega_2^2) &= \omega_7 - \frac{3}{5}\omega_5 \\
h_*(\omega_3^2) &= \omega_8 - \frac{2}{5}\omega_5 \\
h_*(\omega_4^2) &= -\frac{1}{5}\omega_5
\end{aligned}$$

It follows that the set  $H_u$  consists of four elements

$$\omega_1^1 \oplus \omega_2^2, \omega_2^1 \oplus \omega_4^2, \omega_3^1 \oplus \omega_1^2 \text{ and } \omega_4^1 \oplus \omega_3^2$$

whose deficiencies in the group  $SU(9)$  are all 5.

Consequently,  $\ker \pi = \mathbb{Z}_5$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_2^2)$  by (4.3).

vi) If  $u = \frac{\omega_6}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $Spin(10) \times SU(4)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{Spin(10)} \sqcup \Pi_{SU(4)}$  by the weights of  $E_8$ :

$$\begin{aligned} h_*(\omega_1^1) &= \omega_1 - \frac{1}{2}\omega_6 \\ h_*(\omega_4^1) &= \omega_2 - \frac{3}{4}\omega_6 \\ h_*(\omega_5^1) &= \omega_5 - \frac{5}{4}\omega_6 \\ h_*(\omega_1^2) &= \omega_7 - \frac{3}{4}\omega_6 \\ h_*(\omega_2^2) &= \omega_8 - \frac{1}{2}\omega_6 \\ h_*(\omega_3^2) &= -\frac{1}{4}\omega_6 \end{aligned}$$

It follows that the set  $H_u$  consists of 3 elements  $\omega_1^1 \oplus \omega_2^2, \omega_4^1 \oplus \omega_3^2, \omega_5^1 \oplus \omega_1^2$  in which  $\omega_4^1 + \omega_3^2$  has deficiency 4 in the group  $Spin(10) \times SU(4)$ .

Consequently,  $\ker \pi = \mathbb{Z}_4$  with generator  $\exp_1(\omega_4^1) \times \exp_2(\omega_3^2)$  by (4.3).

vii) If  $u = \frac{\omega_7}{3}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $E_6 \times SU(3)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}^s$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1\} \amalg \{\omega_1^2, \omega_2^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Omega = \Pi_{E_6} \sqcup \Pi_{SU(3)}$  by the weights of  $E_8$ :

$$\begin{aligned} h_*(\omega_1^1) &= \omega_1 - \frac{2}{3}\omega_7, \\ h_*(\omega_6^1) &= \omega_6 - \frac{4}{3}\omega_7, \\ h_*(\omega_1^2) &= \omega_8 - \frac{2}{3}\omega_7, \\ h_*(\omega_2^2) &= -\frac{1}{3}\omega_7. \end{aligned}$$

It follows that the set  $H_u$  consists of the two elements  $\omega_1^1 + \omega_2^2$  and  $\omega_6^1 + \omega_1^2$  whose deficiencies in the group  $E_6 \times SU(3)$  are both 3.

Consequently,  $\ker \pi = \mathbb{Z}_3$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_2^2)$  by (4.3).

viii) If  $u = \frac{\omega_8}{2}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $E_7 \times SU(2)$ . Accordingly, assume that the set of fundamental dominant weights of  $C_{\exp(u)}$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1, \omega_7^1\} \amalg \{\omega_1^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Omega = \Pi_{E_7} \sqcup \Pi_{SU(2)}$  by the weights of  $E_8$ :

$$h_*(\omega_7^1) = \omega_7 - \frac{3}{2}\omega_8, \quad h_*(\omega_1^2) = -\frac{1}{2}\omega_8$$

It follows that the set  $H_u$  consists of a single element  $\omega_7^1 + \omega_1^2$  whose deficiency in the group  $E_7 \times SU(2)$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp_1(\omega_7^1) \times \exp_2(\omega_1^2)$ .  $\square$

**Remark 4.5.** Based on concrete constructions of the 1-connected exceptional Lie groups, Yokota obtained also the isomorphism types of maximal subgroups of maximal rank in the recent book [17]. In comparison, our approach is free of the types of the Lie groups.

In our sequel work [8] certain results of Theorem 4.4 are applied to determine the fixed set of the inverse involution  $G \rightarrow G, g \rightarrow g^{-1}$  on an exceptional simple Lie group  $G$ .  $\square$

### 4.3 The isomorphism types of parabolic subgroups

If  $u \in \Delta$  is a vector with  $\beta(u) < 1$ , the centralizer  $C_{\exp(u)}$  is a *parabolic subgroup* of  $G$  whose isomorphism type depends only on the subset  $I_u \subseteq \{1, \dots, n\}$  by Theorem 2.8. The corresponding homogenous space  $G/C_{\exp(u)}$  is a smooth projective variety, called a *flag manifold* of  $G$  [5, 9, 10, 13, 14].

In the case where the set  $I_u$  is a singleton and  $\beta(u) < 1$ , the vector  $u$  is an interior point on an edge of the cell  $\Delta$  from the origin 0. The homogeneous space  $G/C_{\exp(u)}$  is also known as a *generalized Grassmannian* of  $G$  [9].

Theorem 4.3 is ready to apply to determine the isomorphism types of all parabolic subgroups of in a given 1-connected Lie group  $G$ . This is demonstrated in the proof of the next result.

**Theorem 4.6.** *Let  $G$  be an 1-connected exceptional Lie group. For each  $u \in \Delta$  with  $\beta(u) < 1$  and with  $I_u = \{i\}$  a singleton, the isomorphism type of the centralizer  $C_{\exp(u)}$  is given in Table 4 below:*

$G$	$I_u$	the local type of $C_{\exp(u)}$	$\ker \pi$ (generator of $\ker \pi$ )
$G_2$	$\{1\}$	$SU(2) \times S^1$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{2}\omega_1))$
	$\{2\}$	$SU(2) \times S^1$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{2}\omega_2))$
$F_4$	$\{1\}$	$Spin(7) \times S^1$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{2}\omega_1))$
	$\{2\}$	$SU(2) \times SU(3) \times S^1$	$Z_6(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{5}{6}\omega_2))$
	$\{3\}$	$SU(2) \times SU(3) \times S^1$	$Z_6(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{5}{6}\omega_3))$
	$\{4\}$	$Sp(3) \times S^1$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{2}\omega_4))$
$E_6$	$\{1\}$	$Spin(10) \times S^1$	$Z_4(\exp_1(\omega_1^1) \times \exp_2(-\frac{3}{4}\omega_1))$
	$\{2\}$	$SU(6) \times S^1$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{2}\omega_2))$
	$\{3\}$	$SU(2) \times SU(5) \times S^1$	$Z_{10}(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{9}{10}\omega_3))$
	$\{4\}$	$SU(2) \times SU(3) \times SU(3) \times S^1$	$Z_6(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(\omega_3^2) \times \exp_4(\frac{1}{6}\omega_4))$
	$\{5\}$	$SU(2) \times SU(5) \times S^1$	$Z_{10}(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{21}{10}\omega_5))$
	$\{6\}$	$Spin(10) \times S^1$	$Z_4(\exp_1(\omega_1^1) \times \exp_2(-\frac{9}{4}\omega_6))$
$E_7$	$\{1\}$	$Spin(12) \times S^1$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{2}\omega_1))$
	$\{2\}$	$SU(7) \times S^1$	$Z_7(\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{7}\omega_2))$
	$\{3\}$	$SU(2) \times SU(6) \times S^1$	$Z_6(\exp_1(\omega_1^1) \times \exp_2(\omega_4^2) \times \exp_3(-\frac{5}{6}\omega_3))$
	$\{4\}$	$SU(2) \times SU(3) \times SU(4) \times S^1$	$Z_{12}(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(\omega_1^3) \times \exp_4(-\frac{7}{12}\omega_4))$
	$\{5\}$	$SU(3) \times SU(5) \times S^1$	$Z_{15}(\exp_1(\omega_1^1) \times \exp_2(\omega_2^2) \times \exp_3(-\frac{16}{15}\omega_5))$
	$\{6\}$	$SU(2) \times Spin(10) \times S^1$	$Z_4(\exp_1(\omega_1^1) \times \exp_2(\omega_5^2) \times \exp_3(-\frac{3}{4}\omega_6))$
	$\{7\}$	$E_6 \times S^1$	$Z_3(\exp_1(\omega_1^1) \times \exp_2(-\frac{4}{3}\omega_7))$
$E_8$	$\{1\}$	$Spin(14) \times S^1$	$Z_4(\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{4}\omega_1))$
	$\{2\}$	$SU(8) \times S^1$	$Z_8(\exp_1(\omega_1^1) \times \exp_2(-\frac{3}{8}\omega_2))$
	$\{3\}$	$SU(2) \times SU(7) \times S^1$	$Z_{14}(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{11}{14}\omega_3))$
	$\{4\}$	$SU(2) \times SU(3) \times SU(5) \times S^1$	$Z_{30}(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(\omega_1^3) \times \exp_4(-\frac{11}{30}\omega_4))$
	$\{5\}$	$SU(4) \times SU(5) \times S^1$	$Z_{20}(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{17}{20}\omega_5))$
	$\{6\}$	$Spin(10) \times SU(3) \times S^1$	$Z_{12}(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{7}{12}\omega_6))$
	$\{7\}$	$E_6 \times SU(2) \times S^1$	$Z_6(\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{5}{6}\omega_7))$
	$\{8\}$	$E_7 \times S^1$	$Z_2(\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{2}\omega_8))$

Table 5. The parabolic subgroup corresponding to a weight in an 1-connected exceptional Lie groups

**Proof.** Again, the proof will be divided into five cases in accordance to  $G = G_2, F_4, E_6, E_7$  and  $E_8$ .

Since the isomorphism type of the parabolic subgroup  $C_{\exp(\lambda u_i)}$  with  $u_i \in \mathcal{F}_G$  and  $\lambda \in (0, 1)$  is irrelevant with the parameter  $\lambda$ , we can take  $u = \frac{1}{2}u_i$  as a representative for the case  $I_u = \{i\}$ . With this convention the radical part of



$C_{\exp(u)}$  is simply the circle subgroup  $S^1 = \{\exp(t\omega_i) \in G \mid t \in \mathbb{R}\}$  on  $G$  by Theorem 2.8. As a result we have by Lemma 4.1 that

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{t\omega_i \mid t \in \mathbb{Q}\}.$$

**Case 1.**  $G = G_2$

For  $u = \frac{\omega_1}{4}$  (resp.  $\frac{\omega_2}{4}$ ) the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(2)$  is  $\Omega = \{\omega_1^1\}$ . Applying Lemma 4.1 we get the expression of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)}$  by the simple roots of  $G_2$ :

$$h_*(\omega_1^1) = \frac{\alpha_2}{2} (\equiv \frac{1}{2}\omega_1 \bmod \Lambda_G^r) \text{ (resp. } = \frac{\alpha_1}{2} (\equiv \frac{1}{2}\omega_2 \bmod \Lambda_G^r)),$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_1 \mid \lambda \in \mathbb{Q}\} \text{ (resp. } = \{\lambda\omega_2 \mid \lambda \in \mathbb{Q}\}).$$

It follows that the set  $H_u$  consists of the single element  $\omega_1^1$  whose deficiency in the group  $SU(2) \times S^1$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{2}\omega_1)$  (resp.  $\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{2}\omega_2)$ ).

**Case 2.**  $G = F_4$

i) If  $u = \frac{\omega_1}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $Spin(7) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $Spin(7)$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{Spin(7)} = \{\omega_1^1\}$  by the simple roots of  $F_4$ :

$$h_*(\omega_1^1) = \frac{3}{2}\alpha_2 + 2\alpha_3 + \alpha_4 (\equiv \frac{1}{2}\omega_1 \bmod \Lambda_G^r)$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_1 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of the single element  $\omega_1^1$  whose deficiency in the group  $Spin(7)$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp_1(\omega_1^1) \times \exp_2(-\frac{1}{2}\omega_1)$ .

ii) If  $u = \frac{\omega_2}{8}$  (resp.  $\frac{\omega_3}{4}$ ), the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(3) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(2) \times SU(3)$  is  $\Omega = \{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(3)}$  by the simple roots of  $F_4$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{1}{2}\alpha_1 (\equiv \frac{1}{2}\omega_2 \bmod \Lambda_G^r) \\ h_*(\omega_1^2) &= \frac{2}{3}\alpha_3 + \frac{1}{3}\alpha_4 (\equiv \frac{1}{3}\omega_2 \bmod \Lambda_G^r) \\ h_*(\omega_2^2) &= \frac{1}{3}\alpha_3 + \frac{2}{3}\alpha_4 (\equiv \frac{2}{3}\omega_2 \bmod \Lambda_G^r) \end{aligned}$$

(resp.

$$\begin{aligned}
h_*(\omega_1^1) &= \frac{1}{2}\alpha_4 (\equiv \frac{1}{2}\omega_3 \bmod \Lambda_G^r) \\
h_*(\omega_1^2) &= \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 (\equiv \frac{1}{3}\omega_3 \bmod \Lambda_G^r) \\
h_*(\omega_2^2) &= \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2 (\equiv \frac{2}{3}\omega_3 \bmod \Lambda_G^r)
\end{aligned}$$

and that

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(Q)) = \{\lambda\omega_2 \mid \lambda \in Q\} \text{ (resp. } \{\lambda\omega_3 \mid \lambda \in Q\}).$$

It follows that the set  $H_u$  consists of 5 elements in which the one  $\omega_1^1 \oplus \omega_1^2$  has deficiency 6 in the group  $SU(2) \times SU(3)$ .

Consequently,  $\ker \pi = \mathbb{Z}_6$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{5}{6}\omega_2)$  by (4.3).

iii) If  $u = \frac{\omega_4}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $Sp(3) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $Sp(3)$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{Sp(3)} = \{\omega_3^1\}$  by the simple roots of  $F_4$ :

$$h_*(\omega_3^1) = \frac{1}{2}\alpha_1 + \alpha_2 + \frac{3}{2}\alpha_3 (\equiv \frac{1}{2}\omega_4 \bmod \Lambda_G^r)$$

and that

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(Q)) = \{\lambda\omega_4 \mid \lambda \in Q\}.$$

It follows that the set  $H_u$  consists of a single element  $\omega_3^1$  whose deficiency in the group  $Sp(3)$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp_1(\omega_3^1) \times \exp_2(-\frac{1}{2}\omega_4)$ .

**Case 3.**  $G = E_6$

i) If  $u = \frac{\omega_1}{2}$  (resp.  $\frac{\omega_6}{2}$ ), the local type of the centralizer  $C_{\exp(u)}$  is  $Spin(10) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $Spin(10)$  is  $\{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{Spin(10)} = \{\omega_1^1, \omega_4^1, \omega_5^1\}$  by the simple roots of  $E_6$ :

$$\begin{aligned}
h_*(\omega_1^1) &= \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 (\equiv \frac{3}{2}\omega_1 \bmod \Lambda_G^r) \\
h_*(\omega_4^1) &= \frac{3}{4}\alpha_2 + \frac{5}{4}\alpha_3 + \frac{3}{2}\alpha_4 + \alpha_5 + \frac{1}{2}\alpha_6 (\equiv \frac{3}{4}\omega_1 \bmod \Lambda_G^r) \\
h_*(\omega_5^1) &= \frac{5}{4}\alpha_2 + \frac{3}{4}\alpha_3 + \frac{3}{2}\alpha_4 + \alpha_5 + \frac{1}{2}\alpha_6 (\equiv \frac{9}{4}\omega_1 \bmod \Lambda_G^r)
\end{aligned}$$

(resp.

$$\begin{aligned}
h_*(\omega_1^1) &= \alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3 + \alpha_4 + \frac{1}{2}\alpha_5 (\equiv \frac{3}{2}\omega_6 \bmod \Lambda_G^r) \\
h_*(\omega_4^1) &= \frac{1}{2}\alpha_1 + \frac{5}{4}\alpha_2 + \alpha_3 + \frac{3}{2}\alpha_4 + \frac{3}{4}\alpha_5 (\equiv \frac{9}{4}\omega_6 \bmod \Lambda_G^r) \\
h_*(\omega_5^1) &= \frac{1}{2}\alpha_1 + \frac{3}{4}\alpha_2 + \alpha_3 + \frac{3}{2}\alpha_4 + \frac{5}{4}\alpha_5 (\equiv \frac{3}{4}\omega_6 \bmod \Lambda_G^r)
\end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(Q)) = \{\lambda\omega_1 \mid \lambda \in Q\} \text{ (resp. } \{\lambda\omega_6 \mid \lambda \in Q\})$$

It follows that the set  $H_u$  consists of three elements  $\omega_1^1, \omega_4^1, \omega_5^1$  whose deficiencies in the group  $Spin(10)$  are 2, 4, 4, respectively.

Therefore,  $\ker \pi = \mathbb{Z}_4$  with generator  $\exp_1(\omega_5^1) \times \exp_2(-\frac{9}{4}\omega_1)$  (resp.  $\exp_1(\omega_5^1) \times \exp_2(-\frac{9}{4}\omega_6)$ ) by (4.3).

ii) If  $u = \frac{\omega_2}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(6) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(6)$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(6)}$  by the simple roots of  $E_6$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{5}{6}\alpha_1 + \frac{2}{3}\alpha_3 + \frac{1}{2}\alpha_4 + \frac{1}{3}\alpha_5 + \frac{1}{6}\alpha_6 (\equiv \omega_1 - \frac{1}{2}\omega_2 \bmod \Lambda_G^r) \\ h_*(\omega_2^1) &= \frac{2}{3}\alpha_1 + \frac{4}{3}\alpha_3 + \alpha_4 + \frac{2}{3}\alpha_5 + \frac{1}{3}\alpha_6 (\equiv \omega_3 \bmod \Lambda_G^r) \\ h_*(\omega_3^1) &= \frac{1}{2}\alpha_1 + \alpha_3 + \frac{3}{2}\alpha_4 + \alpha_5 + \frac{1}{2}\alpha_6 (\equiv \frac{1}{2}\omega_2 \bmod \Lambda_G^r) \\ h_*(\omega_4^1) &= \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_3 + \alpha_4 + \frac{4}{3}\alpha_5 + \frac{2}{3}\alpha_6 (\equiv \omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_5^1) &= \frac{1}{6}\alpha_1 + \frac{1}{3}\alpha_3 + \frac{1}{2}\alpha_4 + \frac{2}{3}\alpha_5 + \frac{5}{6}\alpha_6 (\equiv \omega_6 - \frac{1}{2}\omega_2 \bmod \Lambda_G^r) \end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_2 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of a single elements  $\omega_3^1$  whose deficiency in the group  $SU(6)$  is 2.

Therefore,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp_1(\omega_3^1) \times \exp_2(-\frac{1}{2}\omega_2)$ .

iii) If  $u = \frac{\omega_3}{4}$  (resp.  $\frac{\omega_5}{4}$ ), the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(5) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(2) \times SU(5)$  is  $\Omega = \{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(5)}$  by the simple roots of  $E_6$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{1}{2}\alpha_1 (\equiv \frac{3}{2}\omega_3 \bmod \Lambda_G^r) \\ h_*(\omega_1^2) &= \frac{4}{5}\alpha_2 + \frac{3}{5}\alpha_4 + \frac{2}{5}\alpha_5 + \frac{1}{5}\alpha_6 (\equiv \frac{12}{5}\omega_3 \bmod \Lambda_G^r) \\ h_*(\omega_2^2) &= \frac{3}{5}\alpha_2 + \frac{6}{5}\alpha_4 + \frac{4}{5}\alpha_5 + \frac{2}{5}\alpha_6 (\equiv \frac{9}{5}\omega_3 \bmod \Lambda_G^r) \\ h_*(\omega_3^2) &= \frac{2}{5}\alpha_2 + \frac{4}{5}\alpha_4 + \frac{6}{5}\alpha_5 + \frac{3}{5}\alpha_6 (\equiv \frac{6}{5}\omega_3 \bmod \Lambda_G^r) \\ h_*(\omega_4^2) &= \frac{1}{5}\alpha_2 + \frac{2}{5}\alpha_4 + \frac{3}{5}\alpha_5 + \frac{4}{5}\alpha_6 (\equiv \frac{3}{5}\omega_3 \bmod \Lambda_G^r) \end{aligned}$$

(resp.

$$\begin{aligned} h_*(\omega_1^1) &= \frac{1}{2}\alpha_6 (\equiv \frac{3}{2}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_1^2) &= \frac{4}{5}\alpha_1 + \frac{1}{5}\alpha_2 + \frac{3}{5}\alpha_3 + \frac{2}{5}\alpha_4 (\equiv \frac{3}{5}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_2^2) &= \frac{3}{5}\alpha_1 + \frac{2}{5}\alpha_2 + \frac{6}{5}\alpha_3 + \frac{4}{5}\alpha_4 (\equiv \frac{6}{5}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_3^2) &= \frac{2}{5}\alpha_1 + \frac{3}{5}\alpha_2 + \frac{4}{5}\alpha_3 + \frac{6}{5}\alpha_4 (\equiv \frac{9}{5}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_4^2) &= \frac{1}{5}\alpha_1 + \frac{4}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{3}{5}\alpha_4 (\equiv \frac{12}{5}\omega_5 \bmod \Lambda_G^r) \end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(Q)) = \{\lambda\omega_3 \mid \lambda \in Q\} \text{ (resp. } = \{\lambda\omega_5 \mid \lambda \in Q\}).$$

It follows that the set  $H_u$  contains 9 elements in which the one  $\omega_1^1 \oplus \omega_1^2$  has deficiency 10 in the group  $SU(2) \times SU(5)$ .

Consequently,  $\ker \pi = \mathbb{Z}_{10}$  with generator  $\exp(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{9}{10}\omega_3)$  (resp.  $\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{21}{10}\omega_5)$ ) by (4.3).

iv) If  $u = \frac{\omega_4}{6}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(3) \times SU(3) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(2) \times SU(3) \times SU(3)$  is  $\Omega = \{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2\} \amalg \{\omega_1^3, \omega_2^3\}$ . Applying Lemma 4.1, we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(3)} \sqcup \Pi_{SU(3)}$  by simple roots of  $E_6$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{1}{2}\alpha_2 (\equiv \frac{1}{2}\omega_4 \bmod \Lambda_G^r) \\ h_*(\omega_1^2) &= \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_3 (\equiv \omega_1 - \frac{1}{3}\omega_4 \bmod \Lambda_G^r) \\ h_*(\omega_2^2) &= \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_3 (\equiv \omega_3 - \frac{2}{3}\omega_4 \bmod \Lambda_G^r) \\ h_*(\omega_1^3) &= \frac{2}{3}\alpha_5 + \frac{1}{3}\alpha_6 (\equiv \omega_5 - \frac{2}{3}\omega_4 \bmod \Lambda_G^r) \\ h_*(\omega_2^3) &= \frac{1}{3}\alpha_5 + \frac{2}{3}\alpha_6 (\equiv \omega_6 - \frac{1}{3}\omega_4 \bmod \Lambda_G^r) \end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_4 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of the 5 elements

$$\omega_1^1, \omega_1^2 \oplus \omega_2^3, \omega_2^2 \oplus \omega_1^3, \omega_1^1 \oplus \omega_1^2 \oplus \omega_2^3, \omega_1^1 \oplus \omega_2^2 \oplus \omega_1^3$$

whose deficiencies in the group  $SU(2) \times SU(3) \times SU(3)$  are 2, 3, 3, 6, 6, respectively.

Consequently,  $\ker \pi = \mathbb{Z}_6$  with generator  $\exp(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(\omega_2^3) \times \exp_4(\frac{1}{6}\omega_4)$  by (4.3).

**Case 4.**  $G = E_7$

i) If  $u = \frac{\omega_1}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $Spin(12) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $Spin(12)$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{Spin(12)} = \{\omega_1^1, \omega_5^1, \omega_6^1\}$  by the simple roots of  $E_7$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 (\equiv \omega_7 - \frac{1}{2}\omega_1 \bmod \Lambda_G^r) \\ h_*(\omega_5^1) &= \alpha_2 + \frac{3}{2}\alpha_3 + 2\alpha_4 + \frac{3}{2}\alpha_5 + \alpha_6 + \frac{1}{2}\alpha_7 (\equiv \frac{1}{2}\omega_1 \bmod \Lambda_G^r) \\ h_*(\omega_6^1) &= \frac{3}{2}\alpha_2 + \alpha_3 + 2\alpha_4 + \frac{3}{2}\alpha_5 + \alpha_6 + \frac{1}{2}\alpha_7 (\equiv \omega_2 \bmod \Lambda_G^r) \end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_1 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of the single element  $\omega_5^1$  whose deficiency in the group  $Spin(12)$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  generated by  $\exp_1(\omega_5^1) \times \exp_2(-\frac{1}{2}\omega_1)$ .

ii) If  $u = \frac{\omega_2}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(7) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(7)$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(7)}$  by the simple roots of  $E_7$ :

$$\begin{aligned}
h_*(\omega_1^1) &= \frac{6}{7}\alpha_1 + \frac{5}{7}\alpha_3 + \frac{4}{7}\alpha_4 + \frac{3}{7}\alpha_5 + \frac{2}{7}\alpha_6 + \frac{1}{7}\alpha_7 (\equiv \frac{10}{7}\omega_2 \bmod \Lambda_G^r) \\
h_*(\omega_2^1) &= \frac{5}{7}\alpha_1 + \frac{10}{7}\alpha_3 + \frac{8}{7}\alpha_4 + \frac{6}{7}\alpha_5 + \frac{4}{7}\alpha_6 + \frac{2}{7}\alpha_7 (\equiv \frac{6}{7}\omega_2 \bmod \Lambda_G^r) \\
h_*(\omega_3^1) &= \frac{4}{7}\alpha_1 + \frac{8}{7}\alpha_3 + \frac{12}{7}\alpha_4 + \frac{9}{7}\alpha_5 + \frac{6}{7}\alpha_6 + \frac{3}{7}\alpha_7 (\equiv \frac{2}{7}\omega_2 \bmod \Lambda_G^r) \\
h_*(\omega_4^1) &= \frac{3}{7}\alpha_1 + \frac{6}{7}\alpha_3 + \frac{9}{7}\alpha_4 + \frac{12}{7}\alpha_5 + \frac{8}{7}\alpha_6 + \frac{4}{7}\alpha_7 (\equiv \frac{12}{7}\omega_2 \bmod \Lambda_G^r) \\
h_*(\omega_5^1) &= \frac{2}{7}\alpha_1 + \frac{4}{7}\alpha_3 + \frac{6}{7}\alpha_4 + \frac{8}{7}\alpha_5 + \frac{10}{7}\alpha_6 + \frac{5}{7}\alpha_7 (\equiv \frac{8}{7}\omega_2 \bmod \Lambda_G^r) \\
h_*(\omega_6^1) &= \frac{1}{7}\alpha_1 + \frac{2}{7}\alpha_3 + \frac{3}{7}\alpha_4 + \frac{4}{7}\alpha_5 + \frac{5}{7}\alpha_6 + \frac{6}{7}\alpha_7 (\equiv \frac{4}{7}\omega_2 \bmod \Lambda_G^r)
\end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_2 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of 6 elements whose deficiencies in the group  $SU(7)$  are all 7.

Consequently,  $\ker \pi = \mathbb{Z}_7$  with generator  $\exp_1(\omega_3^1) \times \exp_2(-\frac{2}{7}\omega_2)$  by (4.3).

iii) If  $u = \frac{\omega_6}{6}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(6) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(2) \times SU(6)$  is  $\Omega = \{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2, \omega_5^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(6)}$  by the simple roots of  $E_7$ :

$$\begin{aligned}
h_*(\omega_1^1) &= \frac{1}{2}\alpha_1 (\equiv \frac{1}{2}\omega_3 \bmod \Lambda_G^r) \\
h_*(\omega_1^2) &= \frac{5}{6}\alpha_2 + \frac{2}{3}\alpha_4 + \frac{1}{2}\alpha_5 + \frac{1}{3}\alpha_6 + \frac{1}{6}\alpha_7 (\equiv \omega_2 - \frac{2}{3}\omega_3 \bmod \Lambda_G^r) \\
h_*(\omega_2^2) &= \frac{2}{3}\alpha_2 + \frac{4}{3}\alpha_4 + \alpha_5 + \frac{2}{3}\alpha_6 + \frac{1}{3}\alpha_7 (\equiv \frac{2}{3}\omega_3 \bmod \Lambda_G^r) \\
h_*(\omega_3^2) &= \frac{1}{2}\alpha_2 + \alpha_4 + \frac{3}{2}\alpha_5 + \alpha_6 + \frac{1}{2}\alpha_7 (\equiv \omega_5 \bmod \Lambda_G^r) \\
h_*(\omega_4^2) &= \frac{1}{3}\alpha_2 + \frac{2}{3}\alpha_4 + \alpha_5 + \frac{4}{3}\alpha_6 + \frac{2}{3}\alpha_7 (\equiv \frac{1}{3}\omega_3 \bmod \Lambda_G^r) \\
h_*(\omega_5^2) &= \frac{1}{6}\alpha_2 + \frac{1}{3}\alpha_4 + \frac{1}{2}\alpha_5 + \frac{2}{3}\alpha_6 + \frac{5}{6}\alpha_7 (\equiv \omega_7 - \frac{1}{3}\omega_3 \bmod \Lambda_G^r)
\end{aligned}$$

and

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_3 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of 5 elements, in which the deficiency of  $\omega_1^1 \oplus \omega_4^2$  in the group  $SU(2) \times SU(6)$  is 6.

Consequently,  $\ker \pi = \mathbb{Z}_6$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_4^2) \times \exp_3(-\frac{5}{6}\omega_3)$ .

iv) If  $u = \frac{\omega_4}{8}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(3) \times SU(4) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(2) \times SU(3) \times SU(4)$  is  $\Omega = \{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2\} \amalg \{\omega_1^3, \omega_2^3, \omega_3^3\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(3)} \sqcup \Pi_{SU(4)}$  by the simple roots of  $E_7$ :

$$\begin{aligned}
h_*(\omega_1^1) &= \frac{1}{2}\alpha_2 (\equiv \omega_2 - \frac{1}{2}\omega_4 \bmod \Lambda_G^r) \\
h_*(\omega_1^2) &= \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_3 (\equiv \frac{2}{3}\omega_4 \bmod \Lambda_G^r) \\
h_*(\omega_2^2) &= \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_3 (\equiv \frac{1}{3}\omega_4 \bmod \Lambda_G^r) \\
h_*(\omega_1^3) &= \frac{3}{4}\alpha_5 + \frac{1}{2}\alpha_6 + \frac{1}{4}\alpha_7 (\equiv \omega_5 - \frac{3}{4}\omega_4 \bmod \Lambda_G^r) \\
h_*(\omega_2^3) &= \frac{1}{2}\alpha_5 + \alpha_6 + \frac{1}{2}\alpha_7 (\equiv \frac{1}{2}\omega_4 \bmod \Lambda_G^r) \\
h_*(\omega_3^3) &= \frac{1}{4}\alpha_5 + \frac{1}{2}\alpha_6 + \frac{3}{4}\alpha_7 (\equiv \omega_7 - \frac{1}{4}\omega_4 \bmod \Lambda_G^r)
\end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_4 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of 11 elements in which the one  $\omega_1^1 \oplus \omega_1^2 \oplus \omega_1^3$  has deficiency 12 in the group  $SU(2) \times SU(3) \times SU(4)$ .

Consequently,  $\ker \pi = \mathbb{Z}_{12}$  with generators  $\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(\omega_1^3) \times \exp_3(-\frac{7}{12}\omega_4)$  by (4.3).

v) If  $u = \frac{\omega_5}{6}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(3) \times SU(5) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(3) \times SU(5)$  is  $\Omega = \{\omega_1^1, \omega_2^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(3)} \sqcup \Pi_{SU(5)}$  by the simple roots of  $E_7$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{2}{3}\alpha_6 + \frac{1}{3}\alpha_7 (\equiv \frac{4}{3}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_2^1) &= \frac{1}{3}\alpha_6 + \frac{2}{3}\alpha_7 (\equiv \frac{2}{3}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_1^2) &= \frac{4}{5}\alpha_1 + \frac{1}{5}\alpha_2 + \frac{3}{5}\alpha_3 + \frac{2}{5}\alpha_4 (\equiv \frac{8}{5}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_2^2) &= \frac{1}{5}\alpha_1 + \frac{4}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{3}{5}\alpha_4 (\equiv \frac{2}{5}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_3^2) &= \frac{3}{5}\alpha_1 + \frac{2}{5}\alpha_2 + \frac{6}{5}\alpha_3 + \frac{4}{5}\alpha_4 (\equiv \frac{6}{5}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_4^2) &= \frac{2}{5}\alpha_1 + \frac{3}{5}\alpha_2 + \frac{4}{5}\alpha_3 + \frac{6}{5}\alpha_4 (\equiv \frac{4}{5}\omega_5 \bmod \Lambda_G^r) \end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_5 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of 14 elements, in which the deficiency of  $\omega_1^2 + \omega_2^2$  in the group  $SU(3) \times SU(5)$  is 15.

Consequently,  $\ker \pi = \mathbb{Z}_{15}$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_2^2) \times \exp_3(-\frac{16}{15}\omega_5)$  by (4.3)

vi) If  $u = \frac{\omega_6}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times Spin(10) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(2) \times Spin(10)$  is  $\Omega = \{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2, \omega_5^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{Spin(10)}$  by the simple roots of  $E_7$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{1}{2}\alpha_7 (\equiv \omega_7 - \frac{1}{2}\omega_6 \bmod \Lambda_G^r) \\ h_*(\omega_1^2) &= \alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3 + \alpha_4 + \frac{1}{2}\alpha_5 (\equiv \frac{1}{2}\omega_6 \bmod \Lambda_G^r) \\ h_*(\omega_4^2) &= \frac{1}{2}\alpha_1 + \frac{5}{4}\alpha_2 + \alpha_3 + \frac{3}{2}\alpha_4 + \frac{3}{4}\alpha_5 (\equiv \omega_2 - \frac{3}{4}\omega_6 \bmod \Lambda_G^r) \\ h_*(\omega_5^2) &= \frac{1}{2}\alpha_1 + \frac{3}{4}\alpha_2 + \alpha_3 + \frac{3}{2}\alpha_4 + \frac{5}{4}\alpha_5 (\equiv \omega_5 - \frac{5}{4}\omega_6 \bmod \Lambda_G^r) \end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_6 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of the three elements

$$\omega_1^2, \omega_1^1 \oplus \omega_4^2, \omega_1^1 \oplus \omega_5^2$$

in which the one  $\omega_1^1 \oplus \omega_5^2$  has deficiency 4 in the group  $SU(2) \times Spin(10)$ .

Consequently,  $\ker \pi = \mathbb{Z}_4$  with generator  $\exp_1(\omega_1^1) \otimes \exp_2(\omega_5^2) \times \exp_3(-\frac{3}{4}\omega_6)$  by (4.3).

vii) If  $u = \frac{\omega_7}{2}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $E_6 \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $E_6$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{E_6} = \{\omega_1^1, \omega_6^1\}$  by the simple roots of  $E_7$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) (\equiv \frac{4}{3}\omega_7 \bmod \Lambda_G^r) \\ h_*(\omega_6^1) &= \frac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6) (\equiv \frac{2}{3}\omega_7 \bmod \Lambda_G^r) \end{aligned}$$

and

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_7 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of the two elements  $\omega_1^1$  and  $\omega_6^1$  whose deficiencies in the group  $E_6$  are both 3.

Consequently  $\ker \pi = \mathbb{Z}_3$  with generator  $\exp_1(\omega_1^1) \times \exp_2(-\frac{4}{3}\omega_7)$  by (4.3).

**Case 5.**  $G = E_8$

i) If  $u = \frac{\omega_1}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $Spin(14) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $Spin(14)$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1, \omega_7^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{Spin(14)} = \{\omega_1^1, \omega_6^1, \omega_7^1\}$  by the simple roots of  $E_8$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 (\equiv \frac{1}{2}\omega_1 \bmod \Lambda_G^r) \\ h_*(\omega_6^1) &= \frac{5}{4}\alpha_2 + \frac{7}{4}\alpha_3 + \frac{5}{2}\alpha_4 + 2\alpha_5 + \frac{3}{2}\alpha_6 + \alpha_7 + \frac{1}{2}\alpha_8 (\equiv \frac{1}{4}\omega_1 \bmod \Lambda_G^r) \\ h_*(\omega_7^1) &= \frac{7}{4}\alpha_2 + \frac{5}{4}\alpha_3 + \frac{5}{2}\alpha_4 + 2\alpha_5 + \frac{3}{2}\alpha_6 + \alpha_7 + \frac{1}{2}\alpha_8 (\equiv \frac{3}{4}\omega_1 \bmod \Lambda_G^r) \end{aligned}$$

and that

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_1 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of three elements in which the deficiency of the element  $\omega_6^1$  in the group  $Spin(14)$  is 4.

Consequently,  $\ker \pi = \mathbb{Z}_4$  with generator  $\exp_1(\omega_6^1) \times \exp_2(-\frac{1}{4}\omega_1)$  by (4.3).

ii) If  $u = \frac{\omega_2}{6}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(8) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(8)$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1, \omega_7^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(8)}$  by the simple roots of  $E_8$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{7}{8}\alpha_1 + \frac{3}{4}\alpha_3 + \frac{5}{8}\alpha_4 + \frac{1}{2}\alpha_5 + \frac{3}{8}\alpha_6 + \frac{1}{4}\alpha_7 + \frac{1}{8}\alpha_8 (\equiv \frac{3}{8}\omega_2 \bmod \Lambda_G^r) \\ h_*(\omega_2^1) &= \frac{3}{4}\alpha_1 + \frac{3}{2}\alpha_3 + \frac{5}{4}\alpha_4 + \alpha_5 + \frac{3}{4}\alpha_6 + \frac{1}{2}\alpha_7 + \frac{1}{4}\alpha_8 (\equiv \frac{3}{4}\omega_2 \bmod \Lambda_G^r) \\ h_*(\omega_3^1) &= \frac{5}{8}\alpha_1 + \frac{5}{4}\alpha_3 + \frac{15}{8}\alpha_4 + \frac{3}{2}\alpha_5 + \frac{9}{8}\alpha_6 + \frac{3}{4}\alpha_7 + \frac{3}{8}\alpha_8 (\equiv \frac{1}{8}\omega_2 \bmod \Lambda_G^r) \\ h_*(\omega_4^1) &= \frac{1}{2}\alpha_1 + \alpha_3 + \frac{3}{2}\alpha_4 + 2\alpha_5 + \frac{3}{2}\alpha_6 + \alpha_7 + \frac{1}{2}\alpha_8 (\equiv \frac{1}{2}\omega_2 \bmod \Lambda_G^r) \\ h_*(\omega_5^1) &= \frac{3}{8}\alpha_1 + \frac{3}{4}\alpha_3 + \frac{9}{8}\alpha_4 + \frac{3}{2}\alpha_5 + \frac{15}{8}\alpha_6 + \frac{5}{4}\alpha_7 + \frac{5}{8}\alpha_8 (\equiv \frac{7}{8}\omega_2 \bmod \Lambda_G^r) \\ h_*(\omega_6^1) &= \frac{1}{4}\alpha_1 + \frac{1}{2}\alpha_3 + \frac{3}{4}\alpha_4 + \alpha_5 + \frac{5}{4}\alpha_6 + \frac{3}{2}\alpha_7 + \frac{3}{4}\alpha_8 (\equiv \frac{1}{4}\omega_2 \bmod \Lambda_G^r) \\ h_*(\omega_7^1) &= \frac{1}{8}\alpha_1 + \frac{1}{4}\alpha_3 + \frac{3}{8}\alpha_4 + \frac{1}{2}\alpha_5 + \frac{5}{8}\alpha_6 + \frac{3}{4}\alpha_7 + \frac{7}{8}\alpha_8 (\equiv \frac{5}{8}\omega_2 \bmod \Lambda_G^r) \end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_2 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of 7 elements, in which the deficiency of  $\omega_1^1$  in the group  $SU(8)$  is 8.

Consequently,  $\ker \pi = \mathbb{Z}_8$  with generator  $\exp_1(\omega_1^1) \times \exp_2(-\frac{3}{8}\omega_2)$  by (4.3).

iii) If  $u = \frac{\omega_3}{8}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(7) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(2) \times SU(7)$  is  $\Omega = \{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2, \omega_5^2, \omega_6^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(7)}$  by the simple roots of  $E_8$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{1}{2}\alpha_1 (\equiv \frac{1}{2}\omega_3 \bmod \Lambda_G^r) \\ h_*(\omega_1^2) &= \frac{6}{7}\alpha_2 + \frac{5}{7}\alpha_4 + \frac{4}{7}\alpha_5 + \frac{3}{7}\alpha_6 + \frac{2}{7}\alpha_7 + \frac{1}{7}\alpha_8 (\equiv \frac{2}{7}\omega_3 \bmod \Lambda_G^r) \\ h_*(\omega_2^2) &= \frac{5}{7}\alpha_2 + \frac{10}{7}\alpha_4 + \frac{8}{7}\alpha_5 + \frac{6}{7}\alpha_6 + \frac{4}{7}\alpha_7 + \frac{2}{7}\alpha_8 (\equiv \frac{4}{7}\omega_3 \bmod \Lambda_G^r) \\ h_*(\omega_3^2) &= \frac{4}{7}\alpha_2 + \frac{8}{7}\alpha_4 + \frac{12}{7}\alpha_5 + \frac{9}{7}\alpha_6 + \frac{6}{7}\alpha_7 + \frac{3}{7}\alpha_8 (\equiv \frac{6}{7}\omega_3 \bmod \Lambda_G^r) \\ h_*(\omega_4^2) &= \frac{3}{7}\alpha_2 + \frac{6}{7}\alpha_4 + \frac{9}{7}\alpha_5 + \frac{12}{7}\alpha_6 + \frac{8}{7}\alpha_7 + \frac{4}{7}\alpha_8 (\equiv \frac{1}{7}\omega_3 \bmod \Lambda_G^r) \\ h_*(\omega_5^2) &= \frac{2}{7}\alpha_2 + \frac{4}{7}\alpha_4 + \frac{6}{7}\alpha_5 + \frac{8}{7}\alpha_6 + \frac{10}{7}\alpha_7 + \frac{5}{7}\alpha_8 (\equiv \frac{3}{7}\omega_3 \bmod \Lambda_G^r) \\ h_*(\omega_6^2) &= \frac{1}{7}\alpha_2 + \frac{2}{7}\alpha_4 + \frac{3}{7}\alpha_5 + \frac{4}{7}\alpha_6 + \frac{5}{7}\alpha_7 + \frac{6}{7}\alpha_8 (\equiv \frac{5}{7}\omega_3 \bmod \Lambda_G^r) \end{aligned}$$

and

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_3 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of 13 elements, in which the deficiency of  $\omega_1^1 \oplus \omega_1^2$  in the group  $SU(2) \times SU(7)$  is 14.

Consequently,  $\ker \pi = \mathbb{Z}_{14}$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{11}{14}\omega_3)$  by (4.3).

iv) If  $u = \frac{\omega_4}{12}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(2) \times SU(3) \times SU(5) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(2) \times SU(3) \times SU(5)$  is  $\Omega = \{\omega_1^1\} \amalg \{\omega_1^2, \omega_2^2\} \amalg \{\omega_1^3, \omega_2^3, \omega_3^3, \omega_4^3\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(2)} \sqcup \Pi_{SU(3)} \sqcup \Pi_{SU(5)}$  by the simple roots of  $E_8$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{1}{2}\alpha_2 (\equiv \frac{1}{2}\omega_4 \bmod \Lambda_G^r) \\ h_*(\omega_1^2) &= \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_3 (\equiv \frac{2}{3}\omega_4 \bmod \Lambda_G^r) \\ h_*(\omega_2^2) &= \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_3 (\equiv \frac{1}{3}\omega_4 \bmod \Lambda_G^r) \\ h_*(\omega_1^3) &= \frac{4}{5}\alpha_5 + \frac{3}{5}\alpha_6 + \frac{2}{5}\alpha_7 + \frac{1}{5}\alpha_8 (\equiv \frac{1}{5}\omega_4 \bmod \Lambda_G^r) \\ h_*(\omega_2^3) &= \frac{3}{5}\alpha_5 + \frac{6}{5}\alpha_6 + \frac{4}{5}\alpha_7 + \frac{2}{5}\alpha_8 (\equiv \frac{2}{5}\omega_4 \bmod \Lambda_G^r) \\ h_*(\omega_3^3) &= \frac{2}{5}\alpha_5 + \frac{4}{5}\alpha_6 + \frac{6}{5}\alpha_7 + \frac{3}{5}\alpha_8 (\equiv \frac{3}{5}\omega_4 \bmod \Lambda_G^r) \\ h_*(\omega_4^3) &= \frac{1}{5}\alpha_5 + \frac{2}{5}\alpha_6 + \frac{3}{5}\alpha_7 + \frac{4}{5}\alpha_8 (\equiv \frac{4}{5}\omega_4 \bmod \Lambda_G^r) \end{aligned}$$

and get



$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_4 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of 29 elements, in which the deficiency of  $\omega_1^1 \oplus \omega_1^2 \oplus \omega_1^3$  in the group  $SU(2) \times SU(3) \times SU(5)$  is 30.

Consequently,  $\ker \pi = \mathbb{Z}_{30}$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(\omega_1^3) \times \exp_4(-\frac{11}{30}\omega_4)$  by (4.3).

v) If  $u = \frac{\omega_5}{10}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $SU(5) \times SU(4) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $SU(5) \times SU(4)$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1\} \amalg \{\omega_1^2, \omega_2^2, \omega_3^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{SU(5)} \sqcup \Pi_{SU(4)}$  by the simple roots of  $E_8$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{4}{5}\alpha_1 + \frac{1}{5}\alpha_2 + \frac{3}{5}\alpha_3 + \frac{2}{5}\alpha_4 (\equiv \frac{3}{5}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_2^1) &= \frac{3}{5}\alpha_1 + \frac{2}{5}\alpha_2 + \frac{6}{5}\alpha_3 + \frac{4}{5}\alpha_4 (\equiv \frac{1}{5}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_3^1) &= \frac{2}{5}\alpha_1 + \frac{3}{5}\alpha_2 + \frac{4}{5}\alpha_3 + \frac{6}{5}\alpha_4 (\equiv \frac{4}{5}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_4^1) &= \frac{1}{5}\alpha_1 + \frac{4}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{3}{5}\alpha_4 (\equiv \frac{2}{5}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_1^2) &= \frac{3}{4}\alpha_6 + \frac{1}{2}\alpha_7 + \frac{1}{4}\alpha_8 (\equiv \frac{1}{4}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_2^2) &= \frac{1}{2}\alpha_6 + \alpha_7 + \frac{1}{2}\alpha_8 (\equiv \frac{1}{2}\omega_5 \bmod \Lambda_G^r) \\ h_*(\omega_3^2) &= \frac{1}{4}\alpha_6 + \frac{1}{2}\alpha_7 + \frac{3}{4}\alpha_8 (\equiv \frac{3}{4}\omega_5 \bmod \Lambda_G^r) \end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_5 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of 19 elements in which the deficiency of  $\omega_1^1 \oplus \omega_1^2$  in the group  $SU(5) \times SU(4)$  is 20.

Consequently,  $\ker \pi = \mathbb{Z}_{20}$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{17}{20}\omega_5)$  by (4.3).

vi) If  $u = \frac{\omega_6}{8}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $Spin(10) \times SU(3) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $Spin(10) \times SU(3)$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1\} \amalg \{\omega_1^2, \omega_2^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{Spin(10)} \sqcup \Pi_{SU(3)}$  by the simple roots of  $E_8$ :

$$\begin{aligned} h_*(\omega_1^1) &= \alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3 + \alpha_4 + \frac{1}{2}\alpha_5 (\equiv \frac{1}{2}\omega_6 \bmod \Lambda_G^r) \\ h_*(\omega_4^1) &= \frac{1}{2}\alpha_1 + \frac{5}{4}\alpha_2 + \alpha_3 + \frac{3}{2}\alpha_4 + \frac{3}{4}\alpha_5 (\equiv \frac{1}{4}\omega_6 \bmod \Lambda_G^r) \\ h_*(\omega_5^1) &= \frac{1}{2}\alpha_1 + \frac{3}{4}\alpha_2 + \alpha_3 + \frac{3}{2}\alpha_4 + \frac{5}{4}\alpha_5 (\equiv \frac{3}{4}\omega_6 \bmod \Lambda_G^r) \\ h_*(\omega_1^2) &= \frac{2}{3}\alpha_7 + \frac{1}{3}\alpha_8 (\equiv \frac{1}{3}\omega_6 \bmod \Lambda_G^r) \\ h_*(\omega_2^2) &= \frac{1}{3}\alpha_7 + \frac{2}{3}\alpha_8 (\equiv \frac{2}{3}\omega_6 \bmod \Lambda_G^r) \end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_6 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of 11 elements in which the deficiency of  $\omega_4^1 \oplus \omega_1^2$  in the group  $Spin(10) \times SU(3)$  is 12.

Consequently,  $\ker \pi = \mathbb{Z}_{12}$  with generator  $\exp_1(\omega_4^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{7}{12}\omega_6)$  by (4.3).

vii) If  $u = \frac{\omega_7}{6}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $E_6 \times SU(2) \times S^1$ . Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $E_6 \times SU(2)$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1\} \amalg \{\omega_1^2\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{E_6} \sqcup \Pi_{SU(2)}$  by the simple roots of  $E_8$ :

$$\begin{aligned} h_*(\omega_1^1) &= \frac{4}{3}\alpha_1 + \alpha_2 + \frac{5}{3}\alpha_3 + 2\alpha_4 + \frac{4}{3}\alpha_5 + \frac{2}{3}\alpha_6 (\equiv \frac{1}{3}\omega_7 \bmod \Lambda_G^r) \\ h_*(\omega_6^1) &= \frac{2}{3}\alpha_1 + \alpha_2 + \frac{4}{3}\alpha_3 + 2\alpha_4 + \frac{5}{3}\alpha_5 + \frac{4}{3}\alpha_6 (\equiv \frac{2}{3}\omega_7 \bmod \Lambda_G^r) \\ h_*(\omega_1^2) &= \frac{1}{2}\alpha_8 (\equiv \frac{1}{2}\omega_7 \bmod \Lambda_G^r) \end{aligned}$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_7 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of 5 elements, in which the deficiency of  $\omega_1^1 \oplus \omega_1^2$  in the group  $E_6 \times SU(2)$  is 6.

Consequently,  $\ker \pi = \mathbb{Z}_6$  with generator  $\exp_1(\omega_1^1) \times \exp_2(\omega_1^2) \times \exp_3(-\frac{5}{6}\omega_7)$  by (4.3).

viii) If  $u = \frac{\omega_8}{4}$ , the local type of the centralizer  $C_{\exp(u)}$  is  $E_7 \times S^1$  by Theorem 2.8. Accordingly, assume that the set of fundamental dominant weights of the semisimple part  $E_7$  is  $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1, \omega_4^1, \omega_5^1, \omega_6^1, \omega_7^1\}$ . Applying Lemma 4.1 we get the expressions of  $h_*(\omega)$  with  $\omega \in \Pi_{E_7} = \{\omega_7^1\}$  by the simple roots of  $E_8$ :

$$h_*(\omega_7^1) = \alpha_1 + \frac{3}{2}\alpha_2 + 2\alpha_3 + 3\alpha_4 + \frac{5}{2}\alpha_5 + 2\alpha_6 + \frac{3}{2}\alpha_7 (\equiv \frac{1}{2}\omega_8 \bmod \Lambda_G^r)$$

and get

$$h_*(\Lambda_{C_{\exp(u)}^{Rad}}^e(\mathbb{Q})) = \{\lambda\omega_8 \mid \lambda \in \mathbb{Q}\}.$$

It follows that the set  $H_u$  consists of the single element  $\omega_7^1$  whose deficiency in the group  $E_7$  is 2.

Consequently,  $\ker \pi = \mathbb{Z}_2$  with generator  $\exp_1(\omega_7^1) \times \exp_2(-\frac{1}{2}\omega_8)$ .  $\square$

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